

Conductance
and Mixing Time

Lecture 1

Probability

Events, probabilities,
conditional probabilities,
union bound

Random variables,
expected value,
linearity of expectation
Markov's inequality

Algorithms

Approx. 3-SAT,

Big idea: we only want the average

Linearity of Expectation
 $E[X+Y] = E[X] + E[Y]$ (stochastically additive)
"average sum = sum of averages"

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$ ($i=1, \dots, m$)

$E[Y_i] = 7/8$ (1 clause case)

$E[\text{\# clauses satisfied}] = E[\sum_{i=1}^m Y_i] = \sum_{i=1}^m E[Y_i] = 7/8 m$

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$

$Z = \sum_{i=1}^m Y_i = \text{\# clauses satisfied}$

$E[Z] = \frac{1}{2} E[Z | x_1 = \text{true}] + \frac{1}{2} E[Z | x_1 = \text{false}]$

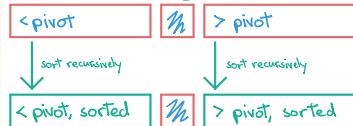
$\Rightarrow$ either $E[Z | x_1 = \text{true}]$ or $E[Z | x_1 = \text{false}]$ is $\geq E[Z]$!

QuickSort,

Random divide and conquer

① pick random 'pivot'

② Divide A into two parts



Expected running time of QuickSort

For each $i, j \in [n]$ w/ $i < j$,

$X_{ij} = \begin{cases} 1 & \text{if } i\text{th smallest element is compared to } j\text{th smallest} \\ 0 & \text{otherwise} \end{cases}$

$\sum_{i < j} X_{ij}$ = total # comparisons made by the algorithm
 $\approx$ total running time

$E[\sum_{i < j} X_{ij}]$ = average running time

$E[\sum_{i < j} X_{ij}] = \sum_{i < j} E[X_{ij}]$ (linearity of expectation)

now analyze $E[X_{ij}]$ in isolation

$E[X_{ij}] = \Pr[X_{ij} = 1]$
 $= \Pr[i\text{th element is compared to } j\text{th}]$

i th and j th elem. selected before any element in between happens w/ probability $\frac{2}{j-i+1}$

$E[X_{ij}] = \frac{2}{j-i+1}$

$E[X_{ij}] = \sum_{i < j} \frac{2}{j-i+1}$ (by linearity of expectation)

$\leq 2n \sum_{k=1}^n \frac{1}{k} \approx 2n \ln n$

$\leq O(n \log n)$ with harmonic series

Quickselect (HW)

Lecture 2

Probability

Hash Functions,

Universal Hash Functions

$h: [n] \rightarrow k$ s.t. for all pairs $i, j \in [n]$,
 $\Pr[h(i) = h(j)] \leq \frac{1}{k}$ (much weaker than ideal hash)

e.g. $h(x) = (ax + b) \bmod p \bmod k$

where • p prime, $> k$

• a in $\{1, \dots, p-1\}$ unif. random

• b in $\{0, \dots, p-1\}$ unif. random



Markov's Inequality

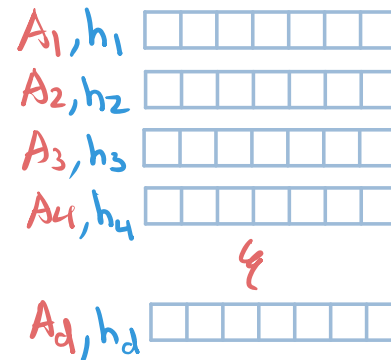
Markov's Inequality (special case)

Let $X \geq 0$ be nonnegative RV

$$\Pr[X \geq 2E[X]] \leq \frac{1}{2}$$

Algorithms

Frequency Estimation, Heavy Hitters (in streams)



estimate f_i w/
 $\min_{j=1, \dots, d} A_j[h_j(i)]$

Lecture 3

Probability

k-wise independence,
higher order moments
(generalizing universal hash
functions and Markov's inequality)

Algorithms

Hash tables w/
chaining +
linear probing

k-wise independent hash functions

$$h: [U] \rightarrow [m]$$

s.t. for any k keys $x_1, x_2, \dots, x_k \in [U]$,
 k values $y_1, y_2, \dots, y_k \in [m]$,

$$P[h(x_1)=y_1, h(x_2)=y_2, \dots, h(x_k)=y_k] = \frac{1}{m^k}$$

eg. $h(x) = a_0 + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1} \pmod{p}$

where $p \geq U$ prime,
 $a_0, a_1, \dots, a_{k-1} \in [p]$ indep, uniformly random

Important lemma

k-wise independent random variables

$$P[X_1=y_1, X_2=y_2, \dots, X_k=y_k] \\ = P[X_1=y_1]P[X_2=y_2] \dots P[X_k=y_k]$$

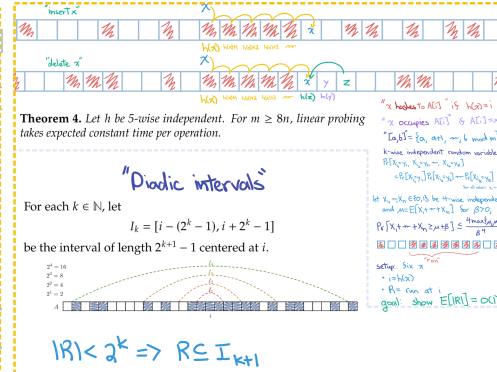
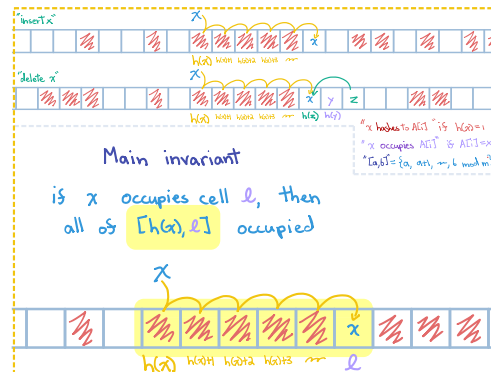
for all values $y_1, \dots, y_k$

let $X_1, \dots, X_n \in \{0, 1\}$ be 4-wise independent
and $\mu = E[X_1 + \dots + X_n]$. then for $\beta > 0$,

$$P[X_1 + \dots + X_n \geq \mu + \beta] \leq \frac{4 \max\{\mu, \mu^3\}}{\beta^4}$$

Main invariant

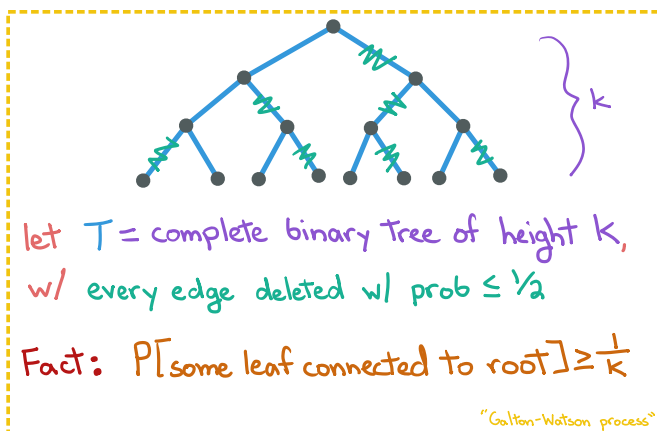
if x occupies cell i , then
all of $[h(x), i]$ occupied



Lecture 4

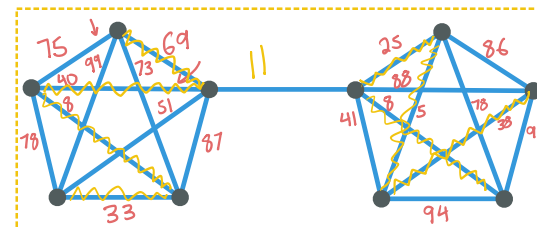
Probability

Galton-Watson processes



Algorithms

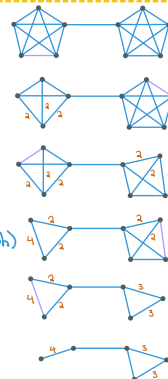
Random Contractions for min cut



Randomized contractions

Input: $G=(V,E)$, weights $c \in \mathbb{R}_{>0}^E$ (unweighted)

- ① while $|E| > 1$
 - Ⓐ sample $e \in E$ in proportion to c
 - Ⓑ "contract" e
- ② return set of edges (in original graph) contracted to remaining edge

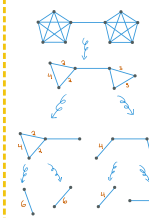


Amplification by branching

Instead of restarting at the beginning, restart partway through

High-level algorithm

- ① randomly contract until prob. of ruining the min-cut is $\sim 1/2$
- ② Branch: make two copies of current graph recurse on both copies



Lecture 5

Probability

Multiplicative Chernoff Bound

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n]$$

$$\epsilon \in (0, 1)$$

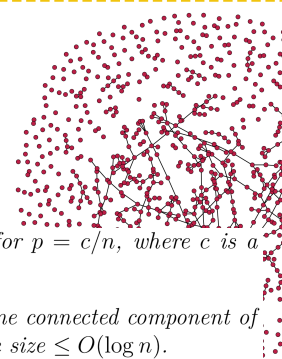
- $P[X_1 + \dots + X_n \geq (1 + \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$

Algorithms (/ results)

Random graphs & the giant component

Erdős-Rényi graphs $G(n, p)$

- n vertices
- each edge sampled independently w/ probability p



Theorem 5.3. Consider a random graph $G \sim \mathcal{G}(n, p)$ for $p = c/n$, where c is a constant.

- If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.
- For $c < 1$, then with high probability, all connected components of G has size $< O(\log n)$.

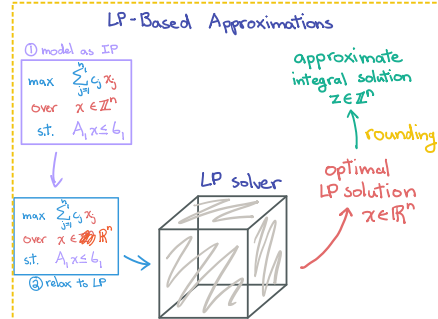
Lecture 6

Probability (& techniques)

Approximations via linear programming & randomized rounding

Linear programs (LPs)
Similar to IPs, except no integrality constraints

min/max $\sum_{j=1}^n c_j x_j$ linear objective
over $x \in \mathbb{R}^n$ linear integral variables
st. $A_1 x \leq b_1$ linear constraints
 $A_2 x = b_2$
 $A_3 x \geq b_3$



Continuous optimization vs Discrete optimization

randomization

Algorithms

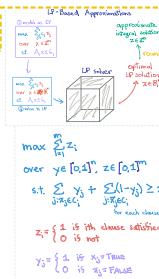
Max Sat Set Cover Covering Integer Programs

For each variable x_j , independently, randomly set $x_j = t$ with probability g_j , and $x_j = f$ otherwise.

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$

k	oblivious	LP	average
1	1/2	$z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
2	3/4	$(3/4)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
3	7/8	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
4	15/16	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
5	31/32	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
...	...	...	...
k	$1 - 1/2^k$	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$

Can we get $\frac{3}{4}$ -APX?



Rounding LP solution
 $x \in [0,1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$

For each set S_j , independently, take S_j with probability x_j .

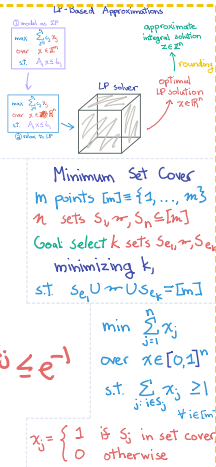
$$E[|F|] = \sum x_j \leq \text{OPT}$$

Fix a point i .

$$E[\# \text{ sets in } F \text{ covering } i] = \sum_{j \in S_i} x_j \geq 1$$

$$P[F \text{ not cover } i] = \prod_{j \in S_i} (1 - x_j) \leq e^{-\sum_{j \in S_i} x_j} \leq e^{-1}$$

$$P[F \text{ cover } i] \geq 1 - e^{-1}$$



Lecture 7

Probability

Continuous Random Variables

Variance, Chebyshev's Variance Reduction

Median trick

Variance

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Chebyshev's inequality

$$P[|X - E[X]| \geq d] \leq \frac{\text{Var}(X)}{d^2}$$

"median trick", "median of means"

- ① take mean to reduce variance, get constant prob. of error
- ② take median to amplify success prob. exponentially

Algorithms

Distinct Elements



Goal: estimate # of distinct elements in the stream

Suppose you had an ideal hash function

$$h: [m] \rightarrow [0, 1]$$

and the following facts:

let $Y_1, \dots, Y_k \in [0, 1]$ independent and uniformly random

$$X = \min \{Y_1, \dots, Y_k\}$$

$$\text{then } E[X] = \frac{1}{k+1}$$

$$\textcircled{1} X \leftarrow 1$$

② for each element e

$$\textcircled{a} X \leftarrow \min \{X, h(e)\}$$

$$/* E[X] = \frac{1}{k+1}, k = (\text{\# distinct elements}) */$$

$$\textcircled{3} \text{ return } \frac{1}{X} - 1$$

$$\text{recall } E[X] = \frac{1}{k+1}$$

$$\text{goal: show } |X - E[X]| \leq \frac{\epsilon}{k+1}$$

w/ good prob.

$$P[|X - E[X]| \geq \frac{\epsilon}{k+1}]$$

$$\leq \text{Var}(X) \cdot \left(\frac{k+1}{\epsilon}\right)^2 \leftarrow \text{Chebyshev's inequality}$$

where

$$\text{Var}(X) = \frac{k}{(k+1)^2(k+2)} \approx \frac{1}{k^2}$$

Q: how to reduce $\text{Var}(X)$?

we can get $(1 \pm \epsilon)$ -APX

w/ 10% error prob. in $O(\frac{1}{\epsilon^2})$ space

what about high probability?

① make q copies $Y_1, \dots, Y_q$ of

② return median of $Y_1, \dots, Y_q$

Algo correct $\Leftrightarrow \geq \frac{1}{2}$ of Y_i 's correct

let $Z = \# \text{correct } Y_i$'s $E[Z] = \frac{q}{2}$

$$P[Z \leq (1 - \frac{\epsilon}{4})E[Z]] \leq e^{-\left(\frac{\epsilon}{4}\right)^2 E[Z]/2} \leq \frac{1}{n^{O(n)}} \text{ for } q = O(\log n)$$

ideal hash $h: [m] \rightarrow [0, 1]$

let $Y_1, \dots, Y_k \in [0, 1]$

indep. and unifs. rand.

$$X = \min \{Y_1, \dots, Y_k\}$$

$$\text{then } E[X] = \frac{1}{k+1}$$

$$\text{and } E[X^2] = \frac{2}{(k+1)(k+2)}$$

$$\text{Var}(X) = E[(X - E[X])^2]$$

$$X_1, \dots, X_q \text{ indep. copies of } X$$

$$\bar{X} = \frac{1}{q}(X_1 + \dots + X_q)$$

$$\text{Var}(\bar{X}) = \frac{1}{q} \text{Var}(X)$$

$$P[X_1 + \dots + X_q \leq (1 - \epsilon)q] \leq e^{-\epsilon^2 q/4}$$

Lecture 8

Probability

Gaussian random variables

Gaussian random variables

$$x \sim \mathcal{N}(\mu, \sigma^2)$$

mean variance

Def: the nicest, most pleasant, convenient random variable ever

Fact let $x \sim \mathcal{N}(\mu, \sigma^2)$, $d \in \mathbb{R}$.

$$\text{then } x+d \sim \mathcal{N}(\mu+d, \sigma^2)$$

$$dx \sim \mathcal{N}(d\mu, d^2\sigma^2)$$

Fact let $x_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $x_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$$\text{then } x_1 + x_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Fact let $x \sim \mathcal{N}(0, 1)$, $t < \frac{1}{2}$

$$\text{then } E[e^{tx^2}] = \frac{1}{\sqrt{1-2t}}$$

Algorithms

Dimensionality reduction

Bag-of-words

literally any sentence is fine

any ↑ is literally sentence English
five encodes text in high dimensions

Lemma 8.8. Let $A \sim \mathcal{N}^{k \times d}$ be a random matrix where each coordinate A_{ij} is an independently drawn sample from $\mathcal{N}$. Let $\epsilon > 0$ be sufficiently small. For any vector x ,

$$\mathbb{P}\left[(1-\epsilon)\|x\|^2 \leq \frac{1}{k}\|Ax\|^2 \leq (1+\epsilon)\|x\|^2\right] \geq 1 - 2e^{-k/8}$$

$$k \left\{ \left[\frac{1}{k} \|Ax\|^2 \right] \leftarrow \alpha_i \right.$$

$$E[\|Ax\|^2] = \sum_{i=1}^k E[(Ax)_i^2] = k\|x\|^2$$

$$E[(\sum_{i=1}^k A_{ij} x_j)^2] = \|x\|^2$$

$$A_{ij} x_j \sim \mathcal{N}(0, x_j^2)$$

$$\sum_{j=1}^n A_{ij} x_j \sim \mathcal{N}(0, \|x\|^2)$$

$$(Ax) \sim \mathcal{N}^k(0, \|x\|^2)$$

$$\Pr[\|Ax\|^2 \geq k\|x\|^2 \cdot (1+\epsilon)] \leq e^{-\epsilon^2 k/8}$$

Dimensionality Reduction - high level

$O(\log n)$ dimensions suffice

where $n = \# \text{ points}$

$$\mathbb{R}^{\text{very large}} \xrightarrow{f(x)} \mathbb{R}^{O(\log n)}$$

One can map n points into $\mathbb{R}^{O(\log n/\epsilon^2)}$ while preserving all distances up to $(1 \pm \epsilon)$ -APX

Lecture 9

Probability

Locality-sensitive hashing

Fix $\theta \in [0, \pi]$. given q , either:

(a) output $p \in P$ w/ $\angle(p, q) \leq 4\theta$

(b) declare no $p \in P$ w/ $\angle(p, q) \leq \theta$

define $h: S^{d-1} \rightarrow \{-1, 1\}^k$ by

$$h_i(x) = \text{sign}(\langle g_i, x \rangle) \text{ w/ } g_i \sim N^d$$

$$k = \pi(\log(n)/2\theta)$$



d-dimensional sphere

when querying y :

- if there is $x \in P$ w/ $\angle(x, y) \leq \theta$, then it collides w/ probability $\geq 1/\sqrt{n}$

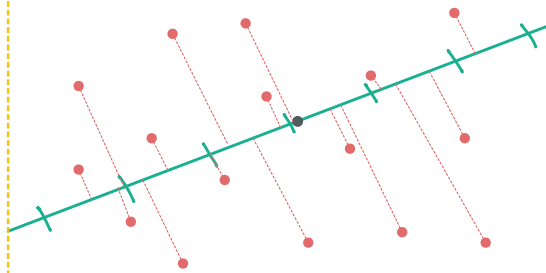
- expect to collide w/ at most one point $x \in P$ w/ $\angle(x, y) > 2\theta$

Lemma let $x, y \in S^{d-1}$
then $g \sim N^d$

$$P[\text{sign}(\langle g, x \rangle) \neq \text{sign}(\langle g, y \rangle)] = \frac{\angle(x, y)}{\pi}$$

Random line embedding & Buckets

$$h(x) = \lfloor \langle g, x \rangle + d \rfloor \text{ w/ } g \sim N^d, d \in [0, 1] \text{ unifs. rand.}$$

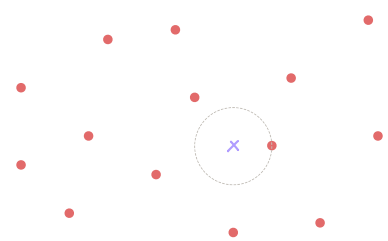


Algorithms

Approximate nearest neighbors

Nearest neighbor search

Input: n points $P \subset \mathbb{R}^d$



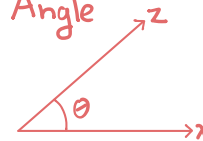
Query: given $q \in \mathbb{R}^d$, find closest point in P to q

Types of distances

- Euclidean distance

$$\|x - z\|_2 = \sqrt{\sum_i (x_i - z_i)^2}$$

- Angle



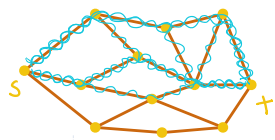
Lecture 10

Probability (and techniques)

LP duality

L_1 -metric embeddings

Max Flow
Min Cut



Path Packing LP

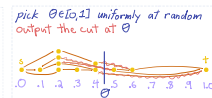
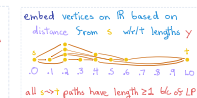
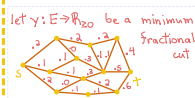
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP
(i.e. cut)

minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

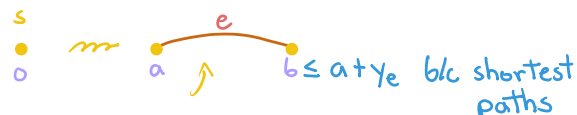
s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$

minimum capacity of any fractional cut

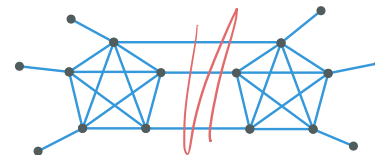


Algorithms

Sparsest Cut

Special case: uniform sparsest cut
w/ uniform demands $b(u,v) = 1 \quad \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S| |T|}$$



useful primitive for graph partitioning, divide & conquer

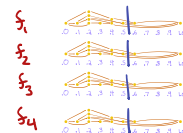
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{i=1}^k b_i d(s_i, t_i) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |f_i(u) - f_i(v)| \quad \text{for } f_i: V \rightarrow [0,1]^k$$

(assume $\min f_i(a) = 0, \max f_i(a) = 1$)



" L_1 -metric"

Lecture 11

Probability (and techniques)

Algorithms

Other metric-based problems
(sometimes via correspondence between cuts $\leftrightarrow$ metrics)

Buy-at-bulk network design
k-median clustering
metric labeling
group Steiner tree
metrical task systems
vehicle routing
hierarchical placement
distributed k-server
covering Steiner tree
⋮



these problems are easy for tree metrics

Approximation by tree metrics

Input: undirected $G=(V,E)$ w/ edge weights $w:E \rightarrow \mathbb{R}_{\geq 0}$

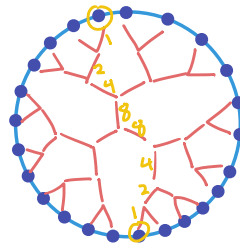
(High-level)
Goal: compute tree T (containing V as nodes)
so that shortest paths on T are close to T

$$d_T(u,v) \geq d_G(u,v)$$

$$\text{For } e=f_u,v \quad \begin{matrix} \text{(distance in } T) \\ \text{(stretch } e) = \frac{d_T(u,v)}{d_G(u,v)} \\ \text{(distance in } G) \end{matrix}$$

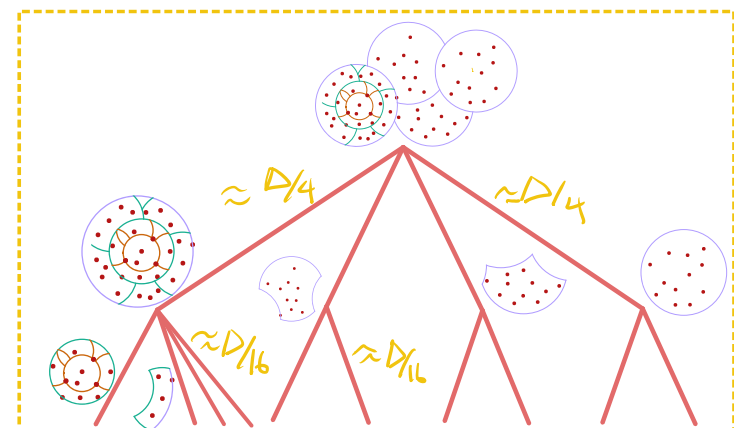
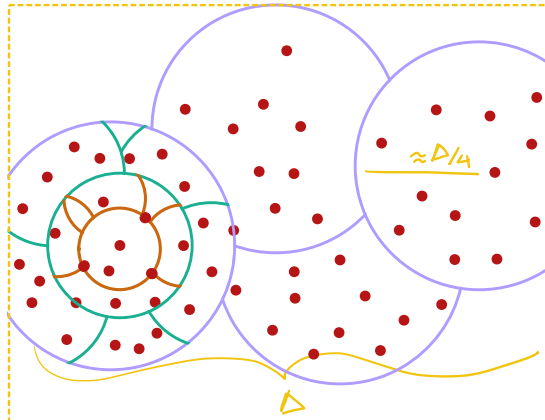
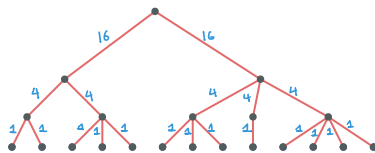
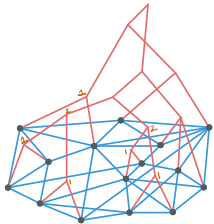
ideally

$$(\text{stretch } e) = \text{polylog}(n) \quad \forall e$$



Randomized Tree metrics

Theorem In randomized polynomial time,
randomized hierarchical dominating tree metric w/
 $E[\text{stretch } e] \leq O(\log n) \quad \forall e \in E$



Lecture 12

Probability (and techniques)

Additive Chernoff Inequality

Double-sampling

VC dimension

Additive Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n] / n$$

$\epsilon \in (0, 1)$

- $P[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon] \leq e^{-2\epsilon^2 n}$
- $P[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon] \leq e^{-2\epsilon^2 n}$

Algorithms

ϵ -samples and ϵ -nets

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -sample theorem [VC71]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample
 ↑ independent of $|P|$ (!)

$P[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon], P[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : \text{asf, lalsn}\}$

$\mu(r) = \frac{|P \cap r|}{|P|}$
 $M_Q(r) = \frac{|Q \cap r|}{|Q|}$
 $RAQ = \{r \cap q : r \in R, q \in Q\}$

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -net theorem [HW87]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -net
 ↑ independent of $|P|$ (!)

$P[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon], P[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : \text{asf, lalsn}\}$

$\mu(r) = \frac{|P \cap r|}{|P|}$
 $M_Q(r) = \frac{|Q \cap r|}{|Q|}$
 $RAQ = \{r \cap q : r \in R, q \in Q\}$

Lecture 13

Probability
(and techniques)

Algorithms
(models and applications)

Consistency model

PAC learning model

Generalization error

Occam's Razor

Generalization by growth

Lecture 14

Probability
(and techniques)

Algorithms
(models and applications)

Entropy

(Optimal)
Error-Correcting
Codes

Lecture 15

Probability (and techniques)

Lovász Local Lemma

Algorithms (models and applications)

Moser's resampling
algorithm

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

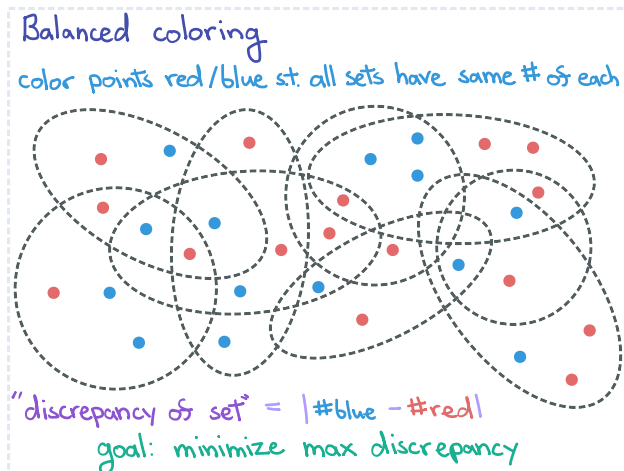
Pagerank

Lecture 17

Probability (and techniques)

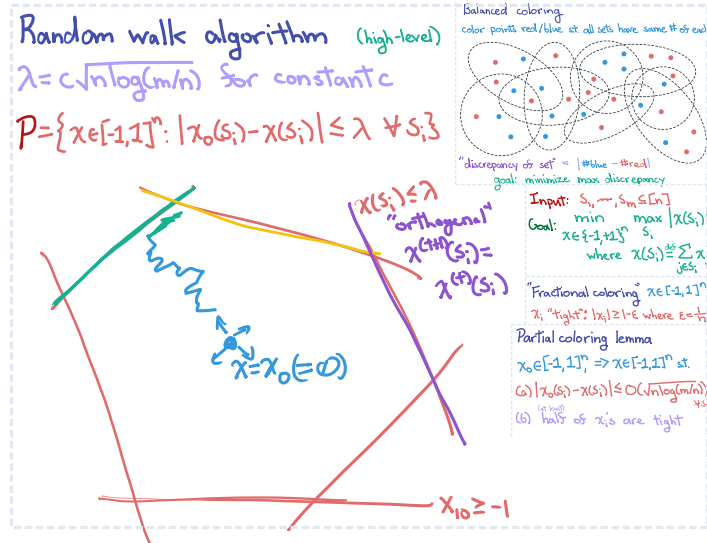
High-dimensional
random walks

More fun w/ Gaussians



Algorithms (models and applications)

Discrepancy
minimization



Lecture 18

Probability (and techniques)

Algorithms (models and applications)

Hitting times

Cover times

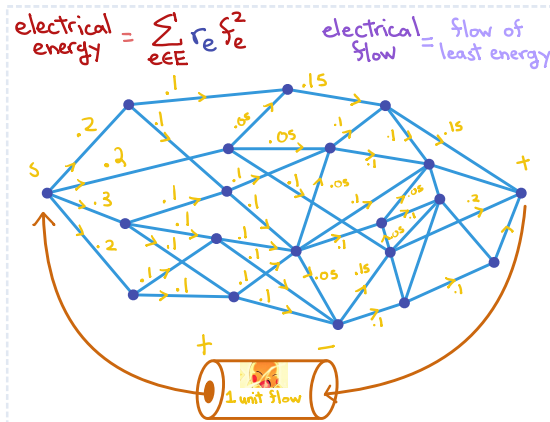
Electrical networks

Connectivity
in $O(\log n)$ space

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

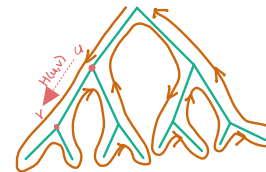
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



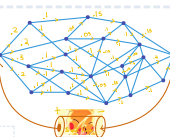
Lemma $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{u, v \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{u, v \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy = $\sum_{e \in E} r_e f_e^2$
electrical flow = least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(BS) $_v$ = net flow at v
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min $\frac{1}{2} \sum_{e \in E} r_e f_e^2$ s.t. $BS = d$
(electrical energy) (is a typical)

effective resistance of d = optimum value/energy
Ohm's law: $f = R^{-1} B^T p$
for electric flow by electric potentials p
 $L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$
(effective resist. of d) $\leq \langle d, L^{-1}d \rangle$

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Lecture 20

Probability
(and techniques)

Algorithms
(models and applications)

Conductance

Cheeger's inequality

Conductance-based
mixing time

Fiedler's algorithm
for low-conductance
cuts

Sparsest cut

Input: undirected graph $G=(V,E)$
edge capacities $c(e) > 0$ for $e \in E$
demands $b(u,v) \geq 0$ for $u,v \in V$ (symmetric)

Goal: compute SCV minimizing

"sparsity" $\frac{c(\delta(S))}{\sum_{u \in S, v \in \bar{S}} b(u,v)}$

← cut few edges

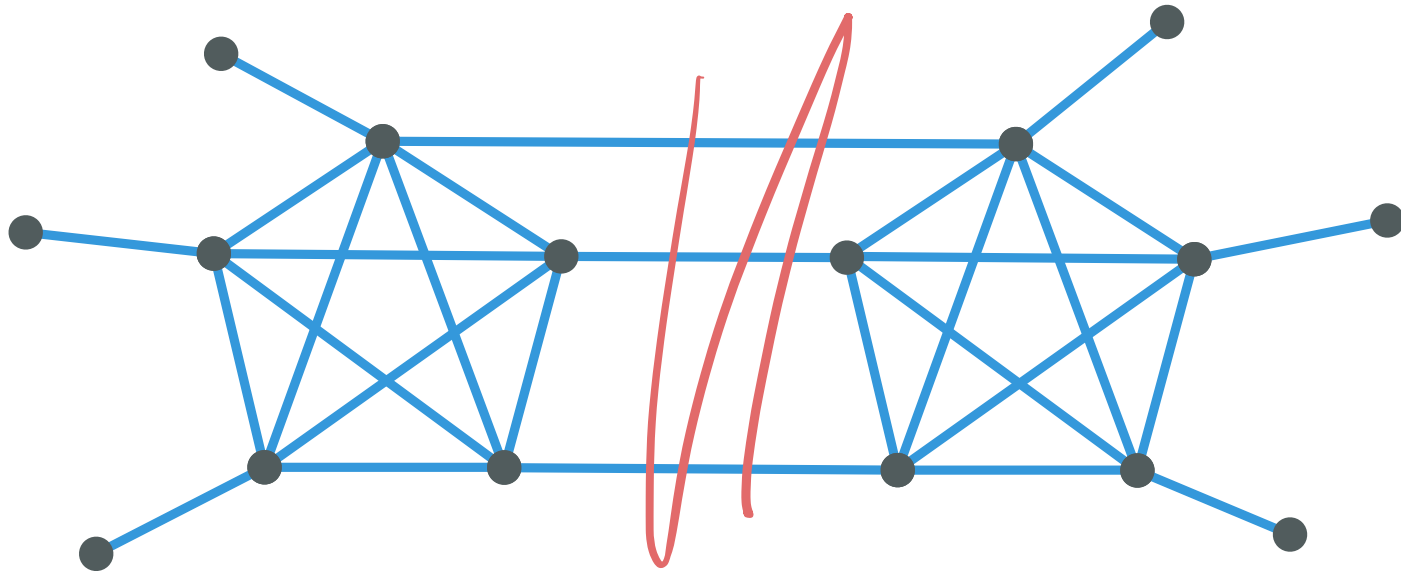
← split up many pairs

" $\delta(S)$ " = edges cut by S

Special case: uniform sparsest cut

w/ uniform demands $b(u,v)=1 \quad \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(S))}{|S| |\bar{S}|}$$



useful primitive for graph partitioning, divide + conquer

Lecture 10

Probability (and techniques)

Algorithms

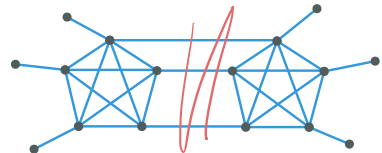
LP duality

L_1 -metric embeddings

Sparsest Cut

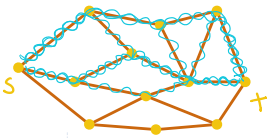
Special case: uniform sparsest cut
w/ uniform demands $b(u,v)=1 \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S||T|}$$



useful primitive for graph partitioning, divide+conquer

Max Flow
Min Cut



Path Packing LP

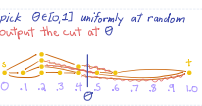
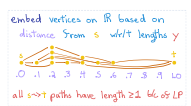
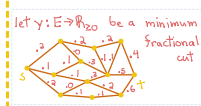
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP
(i.e. cut)

minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$



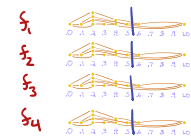
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{\{s,t\}} b(s,t) d(s,t) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |\xi_i(u) - \xi_i(v)| \text{ for } \xi_i: V \rightarrow [0,1]^k$$

(assume $\min \xi_i(a)=1, \max \xi_i(a)=1$)



" L_1 -metric"

Setup / notation

undirected graph $G=(V,E)$

edge weights $w: E \rightarrow \mathbb{R}_{>0}$

for $S \subset V$, " $\delta(S)$ " = set of edges cut by S

" $w(\delta(S))$ " = weight of edges cut by S

for $S \subset V$, " $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$\bar{S} = V \setminus S$

Conductance

$G=(V,E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

"conductance of S "

$$\text{for } S \subset V, \quad \psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

"conductance of G "

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

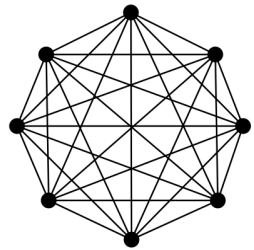
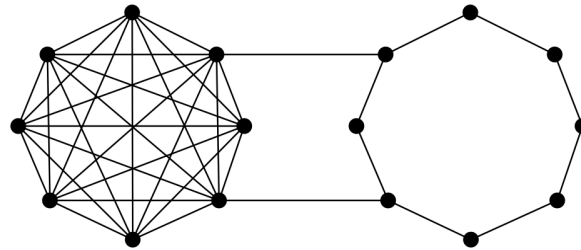
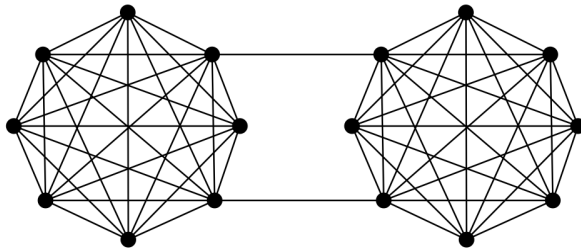
Sparsity

$$\frac{w(\delta(S))}{\min\{|S|, |\bar{S}|\}}$$

Conductance

$$\frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

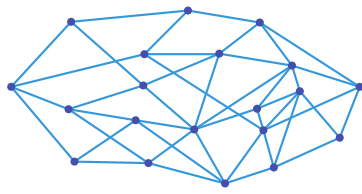
(high/medium/low)



Sparsity

conductance

Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

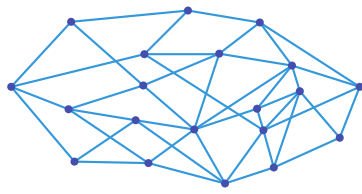
$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

let $D = \text{diag}(d)$ where $d \in \mathbb{R}^V$ are degrees

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$
($1_u = \{0,1\}$ -indicator vector for u)
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

let $D = \text{diag}(d)$ where $d \in \mathbb{R}^V$ are degrees

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
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Laplacian of $e = \{u, v\}$
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$
($1_u = \{0,1\}$ -indicator vector for u)
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

also, $L = D - A$

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

define normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

$$\frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} = \frac{\langle y, My \rangle}{\langle y, y \rangle} \quad \text{for } y = D^{1/2} x$$

$G=(V,E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$D = \text{diag}(\text{degrees})$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Q is symmetric!

Q has same eigenvalues as R !

$$Q = v_1 \otimes v_1 + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

$$\text{w/ } \mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$$

(also eigenvalues of R)

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G = (V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \partial(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1} B C$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary distribution

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

$$M = D^{-1/2} L D^{-1/2}$$

$$= D^{-1/2} (D - A) D^{-1/2}$$

$$= I - Q = D^{-1/2} (I - R) D^{1/2}$$

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subset S \subset V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

$$\checkmark M = I - Q = D^{-1/2} (I - R) D^{1/2}$$

Theorem: M has same eigenvalues as $I - R$.

$$\lambda \text{ for } M \rightarrow \lambda - 1 \text{ for } R$$

eigenvalue eigenvalue

$$\lambda \text{ for } R \rightarrow 1 - \lambda \text{ for } M$$

eigenvalue eigenvalue

e.g. smallest eigenvalue of M is

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M

then
$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S, \bar{S})}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

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Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Theorem: M has same eigenvalues as $I - R$.

$$\lambda \text{ for } M \rightarrow \lambda - 1 \text{ for } R$$

eigenvalue eigenvalue

$$\lambda \text{ for } R \rightarrow 1 - \lambda \text{ for } M$$

eigenvalue eigenvalue

1

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

2

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

3

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

let $\Psi = \Psi(G)$

Lemma: if $\mu_2 \geq |\mu_n|$, then

$$\|R^t x - s\| \leq e^{-t\Psi/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

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↑
stationary
distribution

Q: what if $\mu_2 < |\mu_n|$?

Or: how to force μ_n away from -1 ?

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

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$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

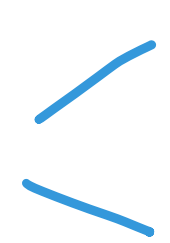
λ for $R \rightarrow 1 - \lambda$ for M
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Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

Lazy random walk

w/ prob. $\frac{1}{2}$:  stay at same vertex
take random step

let $S \in [0,1]^{n \times n}$ be lazy random walk matrix

$$S = \frac{1}{2}I + \frac{1}{2}R$$

note: S has same stationary distribution as R

let $\Psi = \Psi(G)$

Theorem:

$$\|S^\dagger x - s\| \leq e^{-\Psi^2/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
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$$S = \frac{1}{2}I + \frac{1}{2}R$$

Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M
then

exercise

$$\frac{\lambda_2}{2} \leq \underbrace{\Psi(G)}_{\text{algorithmic, rest of class}} \leq \sqrt{2\lambda_2}$$

algorithmic, rest of class

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S)}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

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normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

high level ideas:

$$M = (\lambda_1=0)(v_1 \otimes v_1) + \lambda_2(v_2 \otimes v_2) + \dots + \lambda_n(v_n \otimes v_n)$$

$v_1, \dots, v_n$ orthonormal, $\lambda_1=0 \leq \lambda_2 \leq \dots \leq \lambda_n$

$v_1 =$ eigenvector for eigenvalue 1 of R
 $= \sqrt{d}/\langle \mathbf{1}, d \rangle$

Theorem: M has same eigenvalues as $I-R$.
 λ for $M \rightarrow \lambda-1$ for R
eigenvalue eigenvalue
 λ for $R \rightarrow 1-\lambda$ for M
eigenvalue eigenvalue

$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle y, M y \rangle}{\langle y, y \rangle} \text{ s.t. } \langle y, \sqrt{d} \rangle = 0$$

$$= \min \frac{\langle x, L x \rangle}{\langle x, D x \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

$$M = (\lambda_1=0)(v_1 \otimes v_1) + \lambda_2(v_2 \otimes v_2) + \dots + \lambda_n(v_n \otimes v_n)$$

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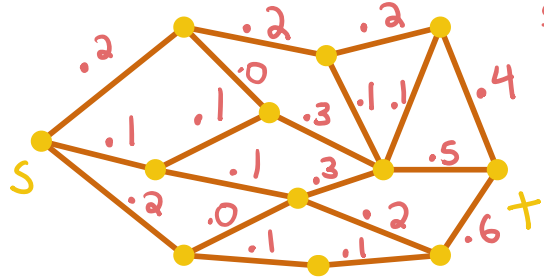
goal: convert x to $S \subseteq V$ s.t.

$$\Psi(S) \leq \sqrt{\frac{\langle x, L x \rangle}{\langle x, D x \rangle}} = \sqrt{\lambda_2}$$

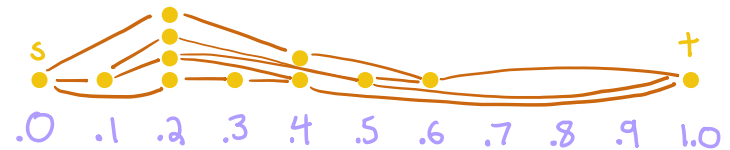
how?

Recall:

let $\gamma: E \rightarrow \mathbb{R}_{\geq 0}$ be a minimum fractional cut



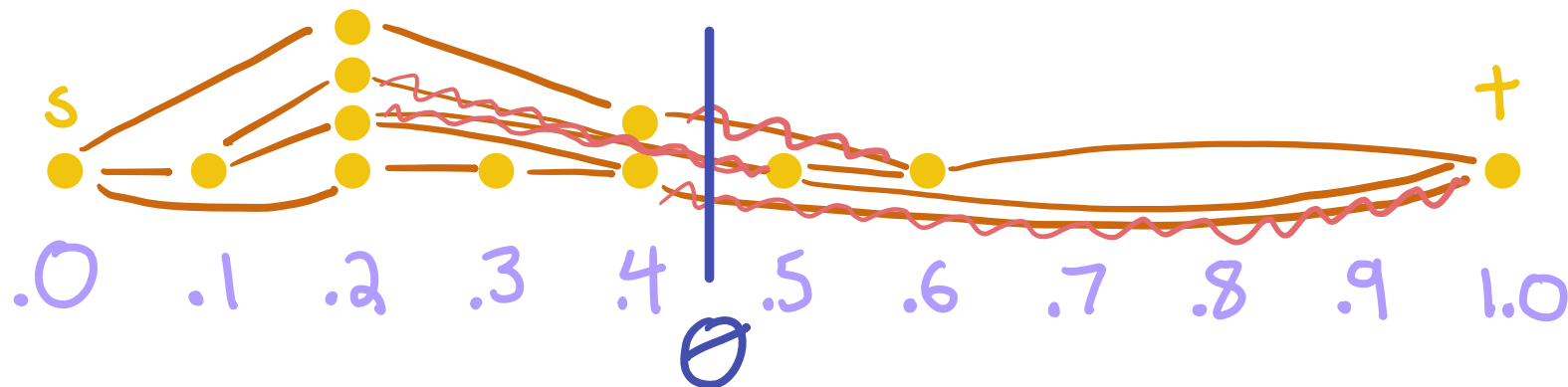
embed vertices on $\mathbb{R}$ based on distance from s w/r/t lengths γ



all $s \rightsquigarrow t$ paths have length ≥ 1 b/c of LP

pick $\theta \in [0, 1]$ uniformly at random

output the cut at θ



$$\Psi(G) \leq \sqrt{2\lambda_2}$$

$$M = (\lambda_1=0)(v_1 \otimes v_1) + \lambda_2(v_2 \otimes v_2) + \dots + \lambda_n(v_n \otimes v_n)$$

$v_1, \dots, v_n$ orthonormal, $\lambda_1=0 \leq \lambda_2 \leq \dots \leq \lambda_n$

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high level ideas:

$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

claim: if $\langle x, d \rangle = 0$, line embedding of x induces

$$\text{cut } S \text{ s.t. } \Psi(S) = \sqrt{2 \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}}$$

let $V = \sum_v \deg(v) = \text{total volume}$

Lemma: let $x \in \mathbb{R}_{\geq 0}^V$ s.t. $\text{vol}(\text{support}(x)) \leq \frac{V}{2}$

then $\langle x, Lx \rangle \geq \frac{\psi^2}{2} \langle x, Dx \rangle$

$$\psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

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$$\text{s.t. } \langle x, d \rangle = 0$$

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$$\text{s.t. } \langle x, d \rangle = 0$$

Lemma: let $\langle x, d \rangle = 0$. then $\forall y = x + d \mathbb{1}$,

$$\frac{\langle y, Ly \rangle}{\langle y, Dy \rangle} \leq \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}$$

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s.t. $\text{vol}(\text{support}(x)) \leq V/2$

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Lemma: let $x = x_+ - x_-$ w/ $x_+, x_- \in \mathbb{R}_{\geq 0}^V$
 then

- $\langle x, Lx \rangle \geq \langle x_+, Lx_+ \rangle + \langle x_-, Lx_- \rangle$
- $\langle x, Dx \rangle \leq \langle x_+, Dx_+ \rangle + \langle x_-, Dx_- \rangle$

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Putting it all together:

$$\text{let } \lambda_2 = \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}, \quad \langle d, x \rangle = 0.$$

$$\psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

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