

Connectivity & electricity

Lecture 1

Probability

Events, probabilities,
conditional probabilities,
union bound

Random variables,
expected value,
linearity of expectation
Markov's inequality

Algorithms

Approx. 3-SAT,

Big idea: we only want the average

Linearity of Expectation
 $E[X+Y] = E[X] + E[Y]$ (stochastically additive)
"average sum = sum of averages"

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$ ($i=1, \dots, m$)

$E[Y_i] = 7/8$ (1 clause case)

$E[\text{\# clauses satisfied}] = E[\sum_{i=1}^m Y_i] = \sum_{i=1}^m E[Y_i] = 7/8 m$

let $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$

$Z = \sum_{i=1}^m Y_i = \text{\# clauses satisfied}$

$E[Z] = \frac{1}{2} E[Z | x_1 = \text{true}] + \frac{1}{2} E[Z | x_1 = \text{false}]$

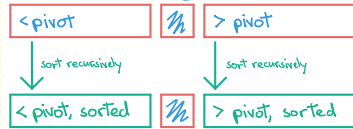
$\Rightarrow$ either $E[Z | x_1 = \text{true}]$ or $E[Z | x_1 = \text{false}]$ is $\geq E[Z]$!

QuickSort,

Random divide and conquer

① pick random 'pivot'

② Divide A into two parts



Expected running time of QuickSort

For each $i, j \in [n]$ w/ $i < j$,

$X_{ij} = \begin{cases} 1 & \text{if } i\text{th smallest element is compared to } j\text{th smallest} \\ 0 & \text{otherwise} \end{cases}$

$\sum_{i < j} X_{ij}$ = total # comparisons made by the algorithm
 $\approx$ total running time

$E[\sum_{i < j} X_{ij}]$ = average running time

$E[\sum_{i < j} X_{ij}] = \sum_{i < j} E[X_{ij}]$ (linearity of expectation, average case is not an average!)

now analyze $E[X_{ij}]$ in isolation

$E[X_{ij}] = \Pr[X_{ij} = 1]$
 $= \Pr[i\text{th element is compared to } j\text{th}]$

i th and j th elem selected before any element in between happens w/ probability $\frac{2}{j-i+1}$

$E[X_{ij}] = \frac{2}{j-i+1}$

$E[X_{ij}] = \sum_{i < j} \frac{2}{j-i+1}$ by linearity of expectation

$\leq 2n \sum_{k=1}^n \frac{1}{k} \approx 2n \ln n$

$\leq O(n \log n)$ with harmonic series

Quickselect (HW)

Lecture 2

Probability

Hash Functions,

Universal Hash Functions

$h: [n] \rightarrow k$ s.t. for all pairs $i, j \in [n]$,
 $\Pr[h(i) = h(j)] \leq \frac{1}{k}$ (much weaker than ideal hash)

e.g. $h(x) = (ax + b) \bmod p \bmod k$

where • p prime, $> k$

• a in $\{1, \dots, p-1\}$ unif. random

• b in $\{0, \dots, p-1\}$ unif. random



Markov's Inequality

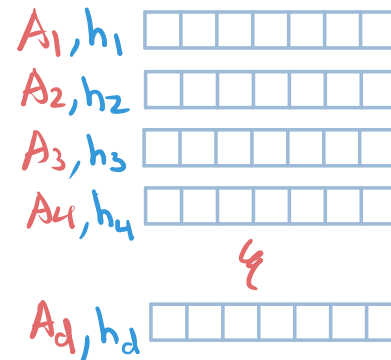
Markov's Inequality (special case)

Let $X \geq 0$ be nonnegative RV

$$\Pr[X \geq 2E[X]] \leq \frac{1}{2}$$

Algorithms

Frequency Estimation, Heavy Hitters (in streams)



estimate f_i w/
 $\min_{j=1, \dots, d} A_j[h_j(i)]$

Lecture 3

Probability

k-wise independence,
higher order moments
(generalizing universal hash
functions and Markov's inequality)

Algorithms

Hash tables w/
chaining +
linear probing

k-wise independent hash functions

$$h: [U] \rightarrow [m]$$

s.t. for any k keys $x_1, x_2, \dots, x_k \in [U]$,
 k values $y_1, y_2, \dots, y_k \in [m]$,

$$P[h(x_1)=y_1, h(x_2)=y_2, \dots, h(x_k)=y_k] = \frac{1}{m^k}$$

eg. $h(x) = a_0 + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1} \pmod{p}$

where $p \geq U$ prime,
 $a_0, a_1, \dots, a_{k-1} \in [p]$ indep, uniformly random

Important lemma

let $X_1, \dots, X_n \in \{0, 1\}$ be 4-wise independent
and $\mu = E[X_1 + \dots + X_n]$. then for $\beta > 0$,

$$P[X_1 + \dots + X_n \geq \mu + \beta] \leq \frac{4 \max\{\mu, \mu^3\}}{\beta^4}$$

k-wise independent random variables

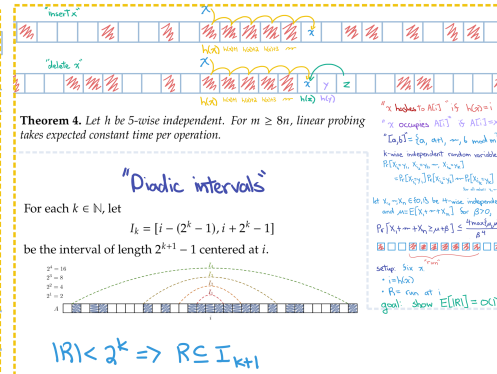
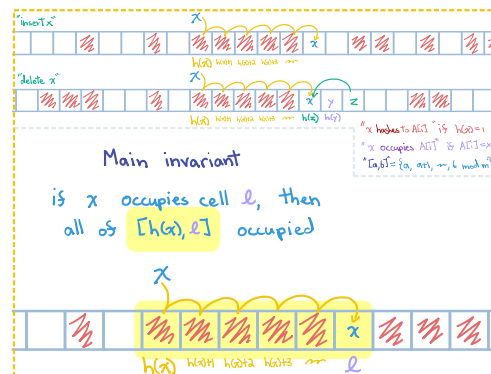
$$P[X_1=y_1, X_2=y_2, \dots, X_k=y_k]$$

$$= P[X_1=y_1] P[X_2=y_2] \dots P[X_k=y_k]$$

for all values $y_1, \dots, y_k$

Main invariant

if x occupies cell i , then
all of $[h(x), i]$ occupied



PLAYER Red
 BADGES 1
 POKÉDEX 10
 TIME 1:24

▶ YES
 NO

Would you like to
 SAVE the game?

Lecture 5

Probability

Multiplicative Chernoff Bound

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n]$$

$$\epsilon \in (0, 1)$$

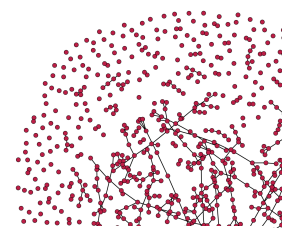
- $P[X_1 + \dots + X_n \geq (1 + \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$

Algorithms (/ results)

Random graphs & the giant component

Erdős-Rényi graphs $G(n, p)$

- n vertices
- each edge sampled independently w/ probability p



Theorem 5.3. Consider a random graph $G \sim \mathcal{G}(n, p)$ for $p = c/n$, where c is a constant.

- If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.
- For $c < 1$, then with high probability, all connected components of G has size $< O(\log n)$.

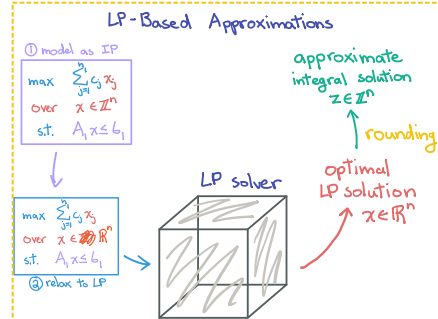
Lecture 6

Probability (& techniques)

Approximations via linear programming & randomized rounding

Linear programs (LPs)
Similar to IPs, except no integrality constraints

min/max $\sum_{j=1}^n c_j x_j$ linear objective
over $x \in \mathbb{R}^n$ linear integral variables
st. $A_1 x \leq b_1$ linear constraints
 $A_2 x = b_2$
 $A_3 x \geq b_3$



Continuous optimization vs Discrete optimization

randomization

Algorithms

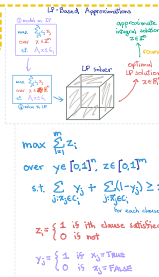
Max Sat Set Cover Covering Integer Programs

For each variable x_j , independently, randomly set $x_j = 1$ with probability p_j , and $x_j = 0$ otherwise.

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$

k	oblivious	LP	average
1	1/2	$z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
2	3/4	$(3/4)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
3	7/8	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
4	15/16	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
5	31/32	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$
...	...	...	...
k	$1 - 1/2^k$	$(1 - 1/e)z_i \geq (3/4)z_i$	$\geq (3/4)z_i$

Can we get $\frac{3}{4}$ -APX?



Rounding LP solution
 $x \in [0, 1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$

For each set S_j , independently, take S_j with probability x_j .

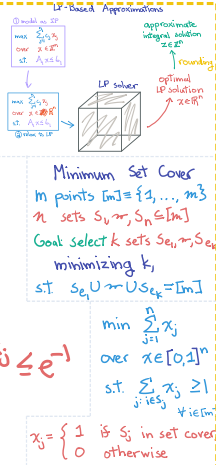
$$E[|F|] = \sum x_j \leq \text{OPT}$$

Fix a point i .

$$E[\# \text{ sets in } F \text{ covering } i] = \sum_{j \in S_i} x_j \geq z_i$$

$$P[F \text{ not cover } i] = \prod_{j \in S_i} (1 - x_j) \leq e^{-\sum_{j \in S_i} x_j} \leq e^{-z_i}$$

$$P[F \text{ cover } i] \geq 1 - e^{-1}$$



Lecture 7

Probability

Continuous Random Variables

Variance, Chebyshev's Variance Reduction

Median trick

Variance

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Chebyshev's inequality

$$P[|X - E[X]| \geq d] \leq \frac{\text{Var}(X)}{d^2}$$

"median trick", "median of means"

- ① take mean to reduce variance, get constant prob. of error
- ② take median to amplify success prob. exponentially

Algorithms

Distinct Elements



Goal: estimate # of distinct elements in the stream

Suppose you had an ideal hash function

$$h: [m] \rightarrow [0, 1]$$

and the following facts:

let $Y_1, \dots, Y_k \in [0, 1]$ independent and uniformly random

$$X = \min \{Y_1, \dots, Y_k\}$$

$$\text{then } E[X] = \frac{1}{k+1}$$

$$\textcircled{1} X \leftarrow 1$$

② for each element e

$$\textcircled{a} X \leftarrow \min \{X, h(e)\}$$

$$/* E[X] = \frac{1}{k+1}, k = (\text{\# distinct elements}) */$$

$$\textcircled{3} \text{ return } \frac{1}{X} - 1$$

$$\text{recall } E[X] = \frac{1}{k+1}$$

$$\text{goal: show } |X - E[X]| \leq \frac{\epsilon}{k+1}$$

w/ good prob.

$$P[|X - E[X]| \geq \frac{\epsilon}{k+1}]$$

$$\leq \text{Var}(X) \cdot \left(\frac{k+1}{\epsilon}\right)^2 \leftarrow \text{Chebyshev's inequality}$$

where

$$\text{Var}(X) = \frac{k}{(k+1)^2(k+2)} \approx \frac{1}{k^2}$$

Q: how to reduce $\text{Var}(X)$?

we can get $(1 \pm \epsilon)$ -APX

w/ 10% error prob. in $O(1/\epsilon^2)$ space

what about high probability?

① make q copies $Y_1, \dots, Y_q$ of

② return median of $Y_1, \dots, Y_q$

Algo correct $\Leftrightarrow \geq \frac{1}{2}$ of Y_i 's correct

let $Z = \# \text{correct } Y_i$'s $E[Z] = \frac{q}{2}$

$$P[Z \leq (1 - \frac{\epsilon}{4})E[Z]] \leq e^{-\left(\frac{\epsilon}{4}\right)^2 E[Z]/2} \leq \frac{1}{n^{O(n)}} \text{ for } q = O(\log n)$$

ideal hash $h: [m] \rightarrow [0, 1]$

let $Y_1, \dots, Y_k \in [0, 1]$

indep. and unifs. rand.

$X = \min \{Y_1, \dots, Y_k\}$

then $E[X] = \frac{1}{k+1}$

and $E[X^2] = \frac{2}{(k+1)(k+2)}$

① $X \leftarrow 1$

② for each element e

③ $X \leftarrow \min \{X, h(e)\}$

$/* E[X] = \frac{1}{k+1}, k = (\text{\# distinct elements}) */$

③ return $\frac{1}{X} - 1$

$\text{Var}(X) = E[(X - E[X])^2]$

$X_1, \dots, X_k$ indep. copies of X

$\bar{X} = \frac{1}{k}(X_1 + \dots + X_k)$

$\text{Var}(\bar{X}) = \frac{1}{k} \text{Var}(X)$

$P[X_1 + \dots + X_k \leq (1 - \epsilon)k \mu]$

$\leq e^{-\epsilon^2 k \mu / 4}$

Lecture 8

Probability

Gaussian random variables

Gaussian random variables

$$x \sim \mathcal{N}(\mu, \sigma^2)$$

mean variance

Def: the nicest, most pleasant, convenient random variable ever

Fact let $x \sim \mathcal{N}(\mu, \sigma^2)$, $d \in \mathbb{R}$.

$$\text{then } x+d \sim \mathcal{N}(\mu+d, \sigma^2)$$

$$dx \sim \mathcal{N}(d\mu, d^2\sigma^2)$$

Fact let $x_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $x_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$$\text{then } x_1 + x_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Fact let $x \sim \mathcal{N}(0, 1)$, $t < \frac{1}{2}$

$$\text{then } E[e^{tx^2}] = \frac{1}{\sqrt{1-2t}}$$

Algorithms

Dimensionality reduction

Bag-of-words

literally any sentence is fine

any ↑ is literally sentence English
five encodes text in high dimensions

Lemma 8.8. Let $A \sim \mathcal{N}^{k \times d}$ be a random matrix where each coordinate A_{ij} is an independently drawn sample from $\mathcal{N}$. Let $\epsilon > 0$ be sufficiently small. For any vector x ,

$$\mathbb{P}\left[(1-\epsilon)\|x\|^2 \leq \frac{1}{k}\|Ax\|^2 \leq (1+\epsilon)\|x\|^2\right] \geq 1 - 2e^{-k/8}.$$

$$k \left\{ \left[\frac{1}{k} \|Ax\|^2 \right] \leftarrow \alpha_i \right.$$

$$E[\|Ax\|^2] = \sum_{i=1}^k E[(Ax)_i^2] = k\|x\|^2$$

$$E[(\sum_{i=1}^k A_{ij} x_j)^2] = \|x\|^2$$

$$A_{ij} x_j \sim \mathcal{N}(0, x_j^2)$$

$$\sum_{j=1}^n A_{ij} x_j \sim \mathcal{N}(0, \|x\|^2)$$

$$(Ax) \sim \mathcal{N}^k(0, \|x\|^2)$$

$$\Pr[\|Ax\|^2 \geq k\|x\|^2 \cdot (1+\epsilon)] \leq e^{-\epsilon^2 k/8}$$

Dimensionality Reduction - high level

$O(\log n)$ dimensions suffice

where $n = \# \text{ points}$

$$\mathbb{R}^{\text{very large}} \xrightarrow{f(x)} \mathbb{R}^{O(\log n)}$$

One can map n points into $\mathbb{R}^{O(\log n/\epsilon^2)}$ while preserving all distances up to $(1 \pm \epsilon)$ -APX

Lecture 9

Probability

Locality-sensitive hashing

Fix $\theta \in [0, \pi]$. given q , either:

(a) output $p \in P$ w/ $\angle(p, q) \leq 4\theta$

(b) declare no $p \in P$ w/ $\angle(p, q) \leq \theta$

define $h: S^{d-1} \rightarrow \{-1, 1\}^k$ by

$$h_i(x) = \text{sign}(\langle g_i, x \rangle) \text{ w/ } g_i \sim N^d$$

$$k = \pi(\log(n)/2\theta)$$



d-dimensional sphere

when querying y :

- if there is $x \in P$ w/ $\angle(x, y) \leq \theta$, then it collides w/ probability $\geq 1/\sqrt{n}$

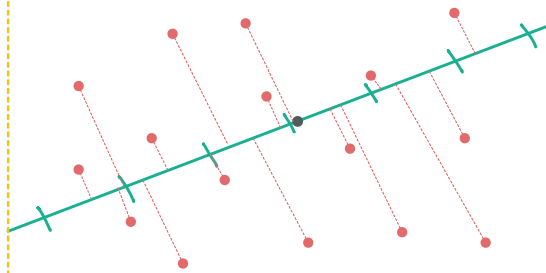
- expect to collide w/ at most one point $x \in P$ w/ $\angle(x, y) > 2\theta$

Lemma let $x, y \in S^{d-1}$
then $g \sim N^d$

$$P[\text{sign}(\langle g, x \rangle) \neq \text{sign}(\langle g, y \rangle)] = \frac{\angle(x, y)}{\pi}$$

Random line embedding & Buckets

$$h(x) = \lfloor \langle g, x \rangle + d \rfloor \text{ w/ } g \sim N^d, d \in [0, 1] \text{ unifs. rand.}$$

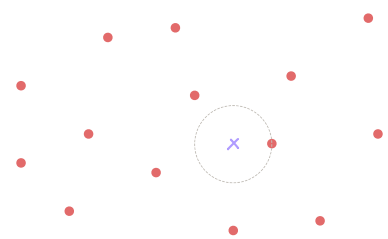


Algorithms

Approximate nearest neighbors

Nearest neighbor search

Input: n points $P \subset \mathbb{R}^d$



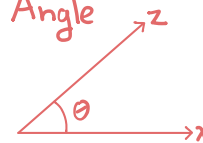
Query: given $q \in \mathbb{R}^d$, find closest point in P to q

Types of distances

- Euclidean distance

$$\|x - z\|_2 = \sqrt{\sum_i (x_i - z_i)^2}$$

- Angle



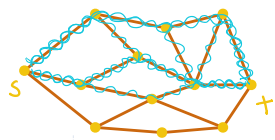
Lecture 10

Probability (and techniques)

LP duality

L_1 -metric embeddings

Max Flow Min Cut



Path Packing LP

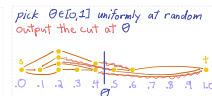
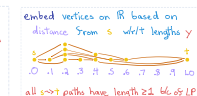
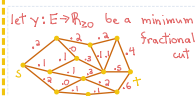
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP (i.e. cut)

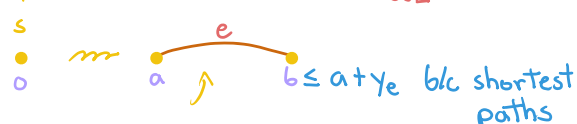
minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$

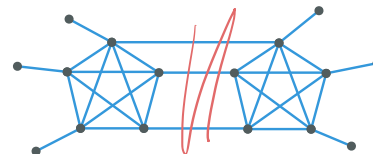


Algorithms

Sparsest Cut

Special case: uniform sparsest cut
w/ uniform demands $b(u,v) = 1 \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S| |T|}$$



useful primitive for graph partitioning, divide & conquer

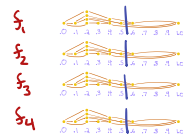
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{\{s,t\}} b(s,t) d(s,t) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |\xi_i(u) - \xi_i(v)| \text{ for } \xi_i: V \rightarrow [0,1]^k$$

(assume $\min \xi_i(a) = 0, \max \xi_i(a) = 1$)



" L_1 -metric"

Sparsest cut
Input: undirected graph $G=(V,E)$,
edge capacities $c(e) > 0$ for $e \in E$,
demands $b(s,t) > 0$ for $s,t \in V$.
Goal: compute SCF minimizing
"sparsity" $\frac{c(\delta(s))}{|S| |T|}$ where S and T are edge cut sets.

Lecture 11

Probability (and techniques)

Algorithms

Other metric-based problems
(sometimes via correspondence between cuts $\leftrightarrow$ metrics)

Buy-at-bulk network design
k-median clustering
metric labeling
group Steiner tree
metrical task systems
vehicle routing
hierarchical placement
distributed k-server
covering Steiner tree
⋮



these problems are easy for tree metrics

Approximation by tree metrics

Input: undirected $G=(V,E)$ w/ edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

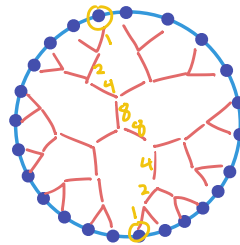
(High-level)
Goal: compute tree T (containing V as nodes)
so that shortest paths on T are close to T

$$d_T(u,v) \geq d_G(u,v)$$

$$\text{For } e=f_u, v \quad \begin{matrix} \text{(distance in } T) \\ \text{(stretch } e) = \frac{d_T(u,v)}{d_G(u,v)} \\ \text{(distance in } G) \end{matrix}$$

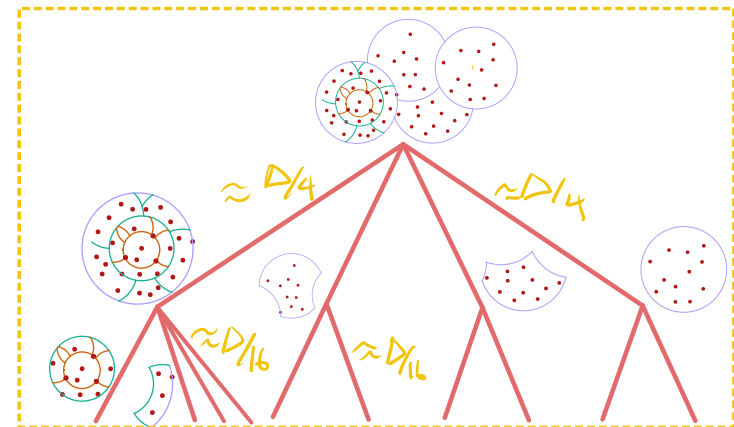
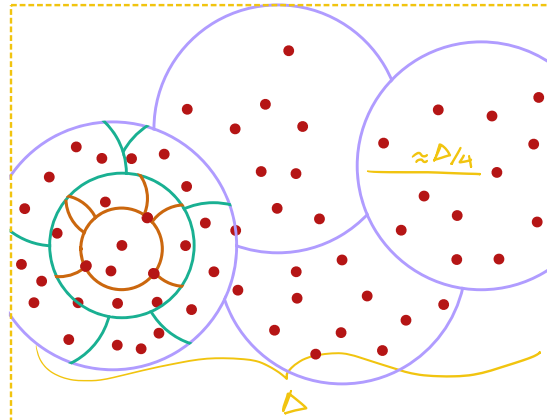
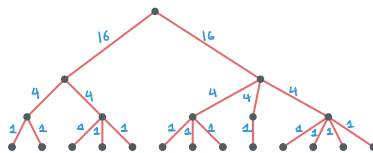
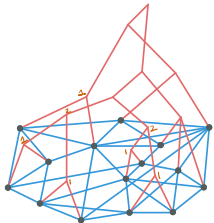
ideally

$$(\text{stretch } e) = \text{polylog}(n) \quad \forall e$$



Randomized Tree metrics

Theorem In randomized polynomial time,
randomized hierarchical dominating tree metric w/
 $E[\text{stretch } e] \leq O(\log n) \quad \forall e \in E$



Lecture 12

Probability (and techniques)

Additive Chernoff Inequality

Double-sampling

VC dimension

Additive Chernoff bound

let $X_1, \dots, X_n \in [0, 1]$ be independent

$$\mu = E[X_1 + \dots + X_n] / n$$

$\epsilon \in (0, 1)$

- $P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right] \leq e^{-2\epsilon^2 n}$
- $P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$

Algorithms

ϵ -samples and ϵ -nets

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -sample theorem [VC71]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample
 ↑ independent of $|P|$ (!)

$P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right], P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : \text{asf, lalsn}\}$

$\mu(r) = \frac{|P \cap r|}{|P|}$
 $M_Q(r) = \frac{|Q \cap r|}{|Q|}$
 $RAQ = \{r \cap q : r \in R, q \in Q\}$

ϵ -sample $|M_Q(r) - \mu(r)| \leq \epsilon \quad \forall r \in R$

let Q sample k points w/ repetition
 $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -sample $k = O\left(\frac{\ln(g(|P|))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

ϵ -net theorem [HW87]
 $k \geq O\left(\frac{\ln(g(2k))}{\epsilon^2}\right) \Rightarrow \epsilon$ -net (w/ constant probability)

e.g. $g(n) = O(n^d)$
 $k \geq O\left(\frac{d \ln(d/\epsilon)}{\epsilon^2}\right) \Rightarrow \epsilon$ -net
 ↑ independent of $|P|$ (!)

$P\left[\frac{X_1 + \dots + X_n}{n} \geq \mu + \epsilon\right], P\left[\frac{X_1 + \dots + X_n}{n} \leq \mu - \epsilon\right] \leq e^{-2\epsilon^2 n}$ $g(n) = \max\{|RAQ| : \text{asf, lalsn}\}$

$\mu(r) = \frac{|P \cap r|}{|P|}$
 $M_Q(r) = \frac{|Q \cap r|}{|Q|}$
 $RAQ = \{r \cap q : r \in R, q \in Q\}$

Lecture 13

Probability
(and techniques)

Algorithms
(models and applications)

Consistency model

PAC learning model

Generalization error

Occam's Razor

Generalization by growth

Lecture 14

Probability
(and techniques)

Algorithms
(models and applications)

Entropy

(Optimal)
Error-Correcting
Codes

Lecture 15

Probability (and techniques)

Lovász Local Lemma

Algorithms (models and applications)

Moser's resampling
algorithm

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

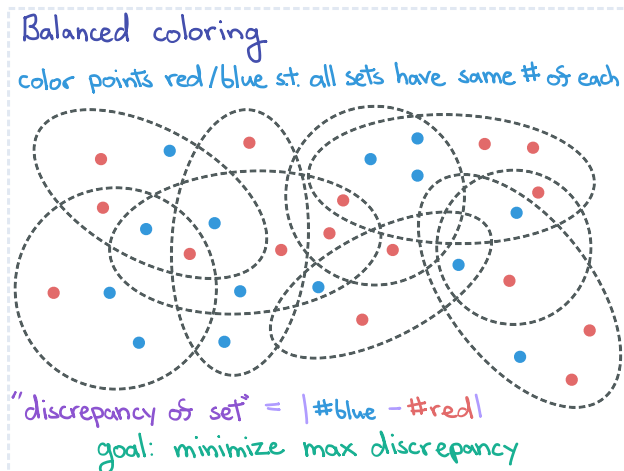
Pagerank

Lecture 17

Probability (and techniques)

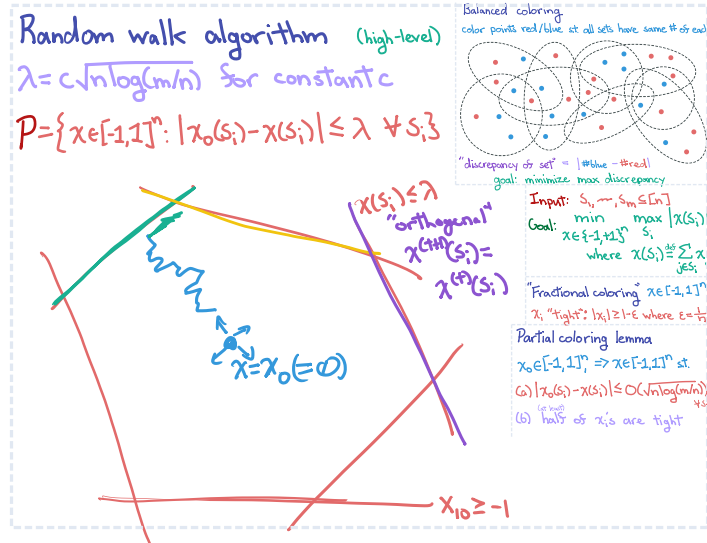
High-dimensional random walks

More fun w/ Gaussians



Algorithms (models and applications)

Discrepancy minimization

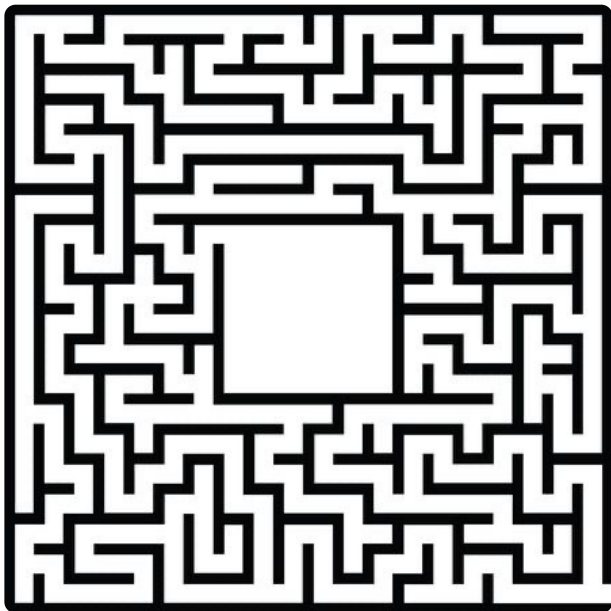


connectivity

Input: $G=(V,E)$ undirected,
(m edges, n vertices)

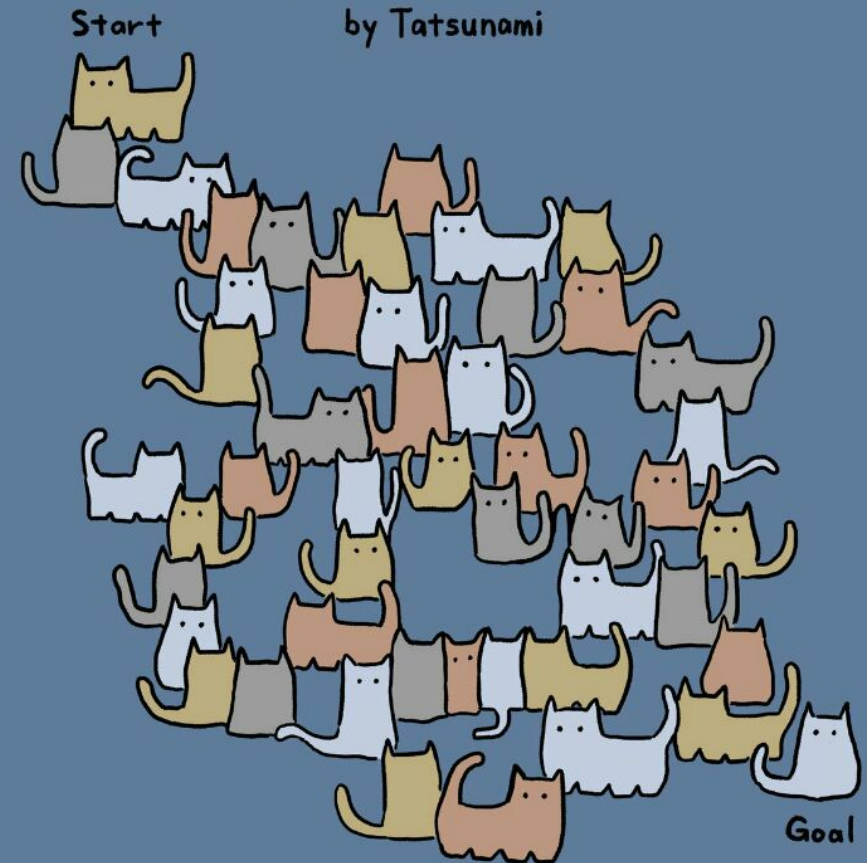
$s, t \in V$

Q: is there a path from
s to t?



CAT MAZE

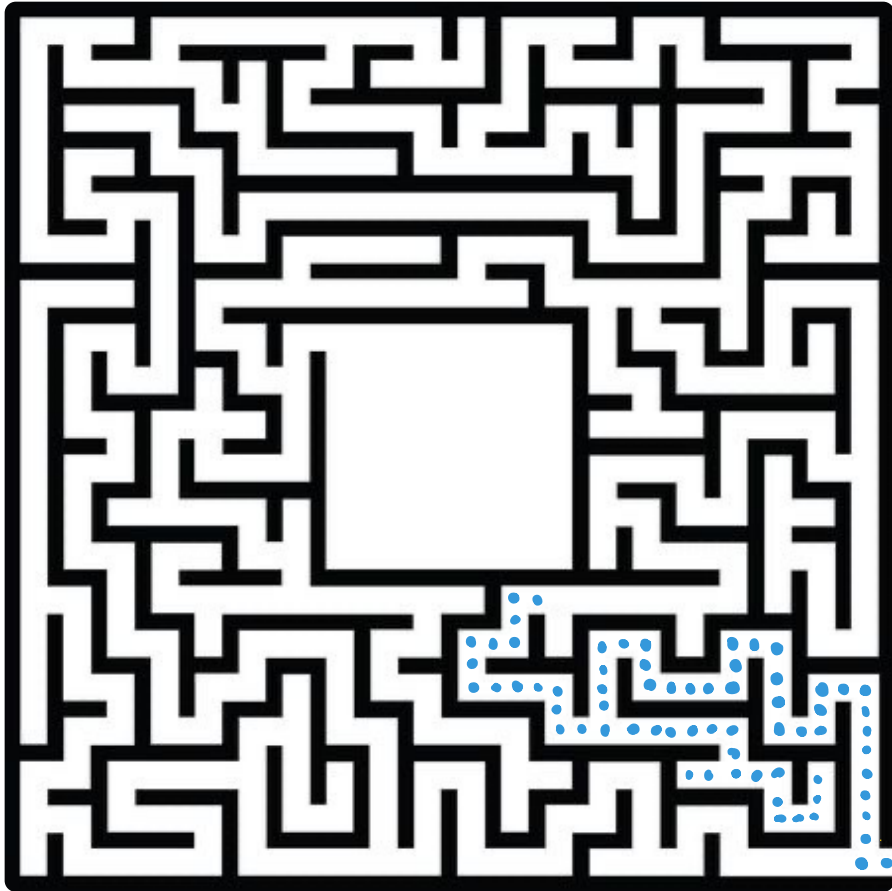
by Tatsunami



Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.



Breadcrumbs

BFS, DFS take
linear time and space

can connectivity be solved in...

Lies

Yes

No

Not
voting

linear space?

BFS,
DFS

$O(\log^{O(1)} n)$ space?

90%

1

3

$O(\log n)$ space?

90%

4

2

$O(\log n)$ space
deterministically?

50%

50%

2 weeks
from now

Lecture 18

Probability
(and techniques)

Hitting times

Cover times

Electrical networks

Algorithms
(models and applications)

Connectivity
in $O(\log n)$ space

can connectivity be solved in $O(\log n)$ space?

algorithm:

(1) repeat $O(\log n)$ times:

(a) start at $s \in V$

(b) randomly walk for $O(mn)$ steps.

(c) if walk visits t , return TRUE

(2) return False

random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

can connectivity be solved in $O(\log n)$ space?

algorithm:

- (1) repeat $O(\log n)$ times:
 - (a) start at $s \in V$
 - (b) randomly walk for $O(mn)$ steps.
 - (c) if walk visits t , return TRUE
- (2) return False

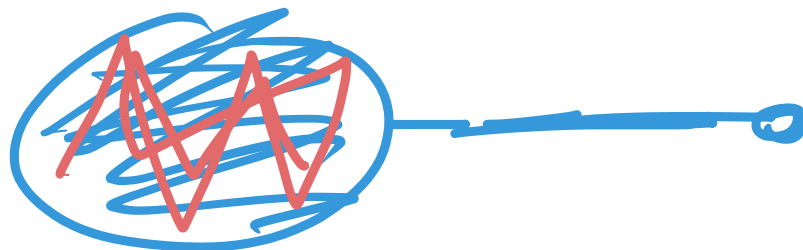
random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

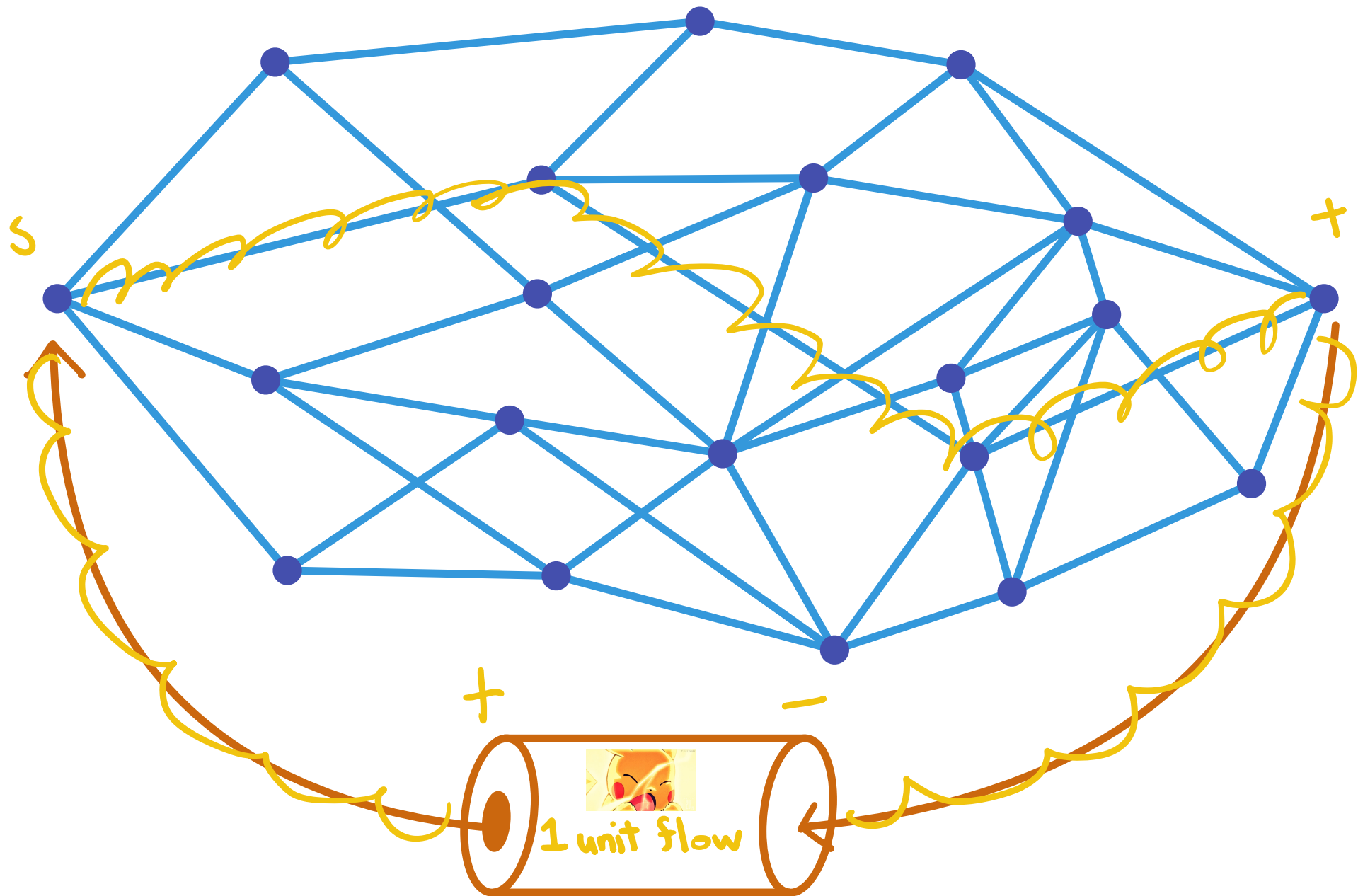
Theorem

If G is connected, then $C(s) \leq m(n-1)$





Electrical networks



electricity takes the "path of least resistance"

"flow of least resistance"

each edge e has resistance $r_e > 0$

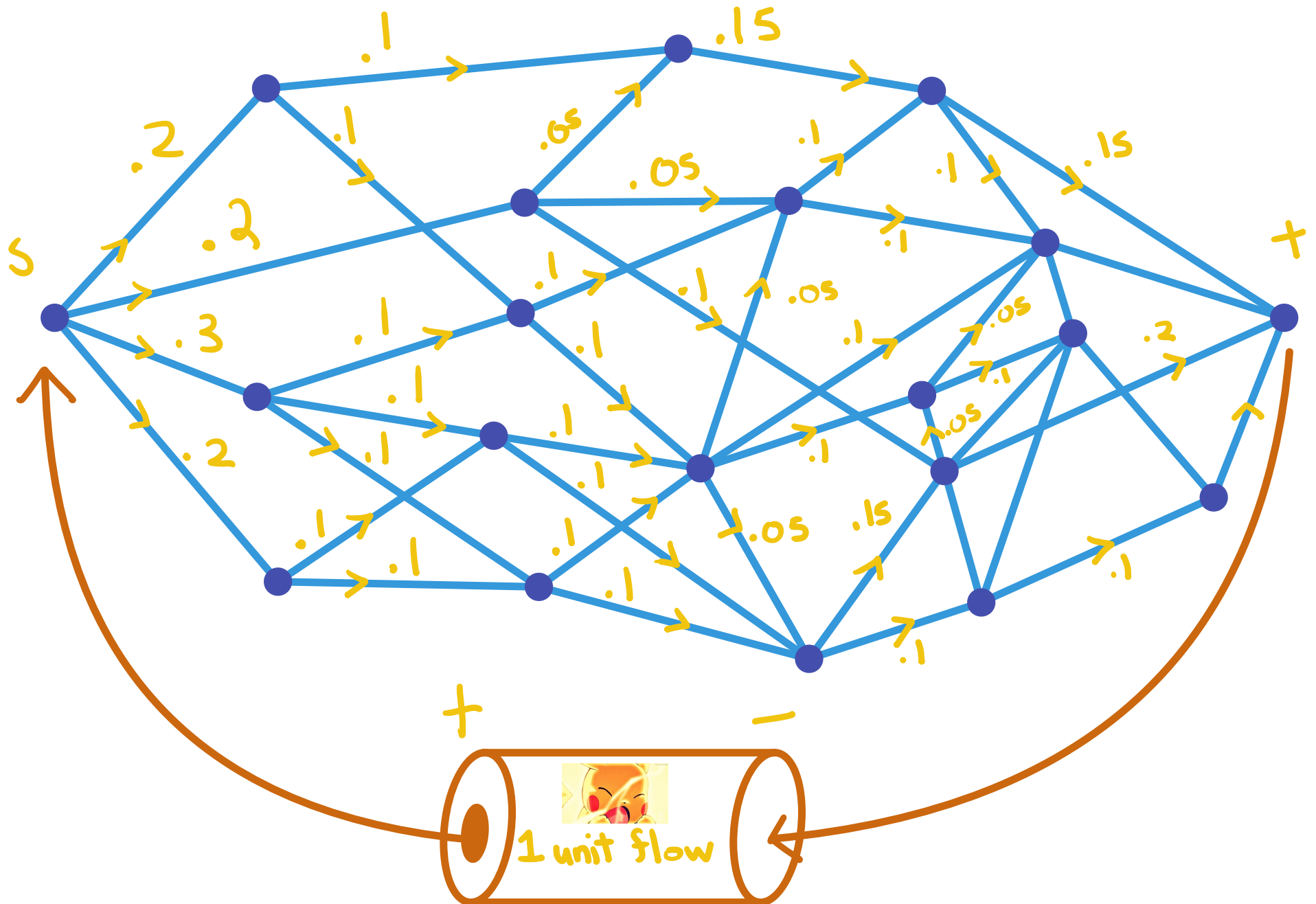
for flow $f: E \rightarrow \mathbb{R}_{\geq 0}$,

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

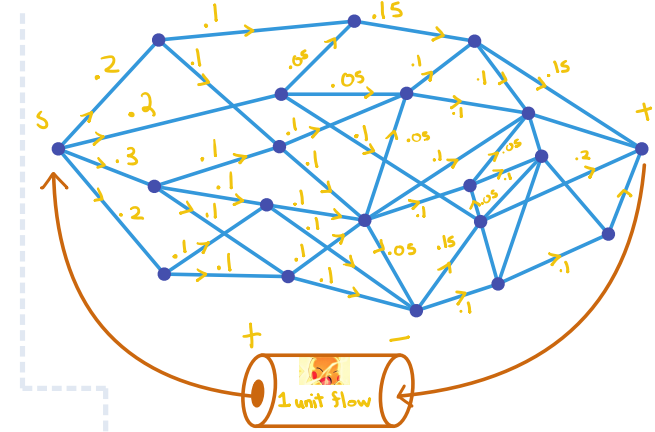
electrical energy = $\sum_{e \in E} r_e f_e^2$

electrical flow = flow of least energy



Electrical flow = convex minimization

(need math setup)



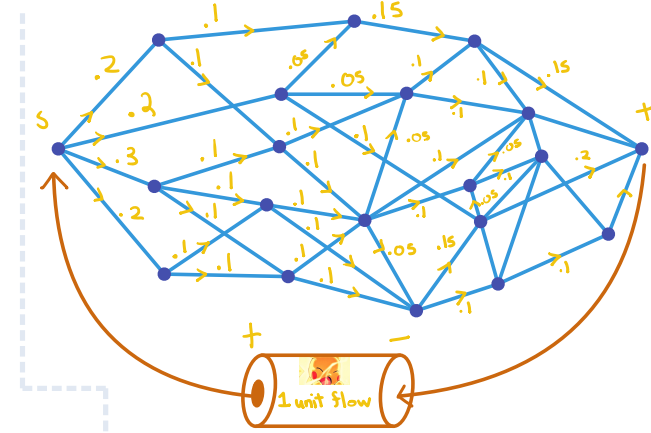
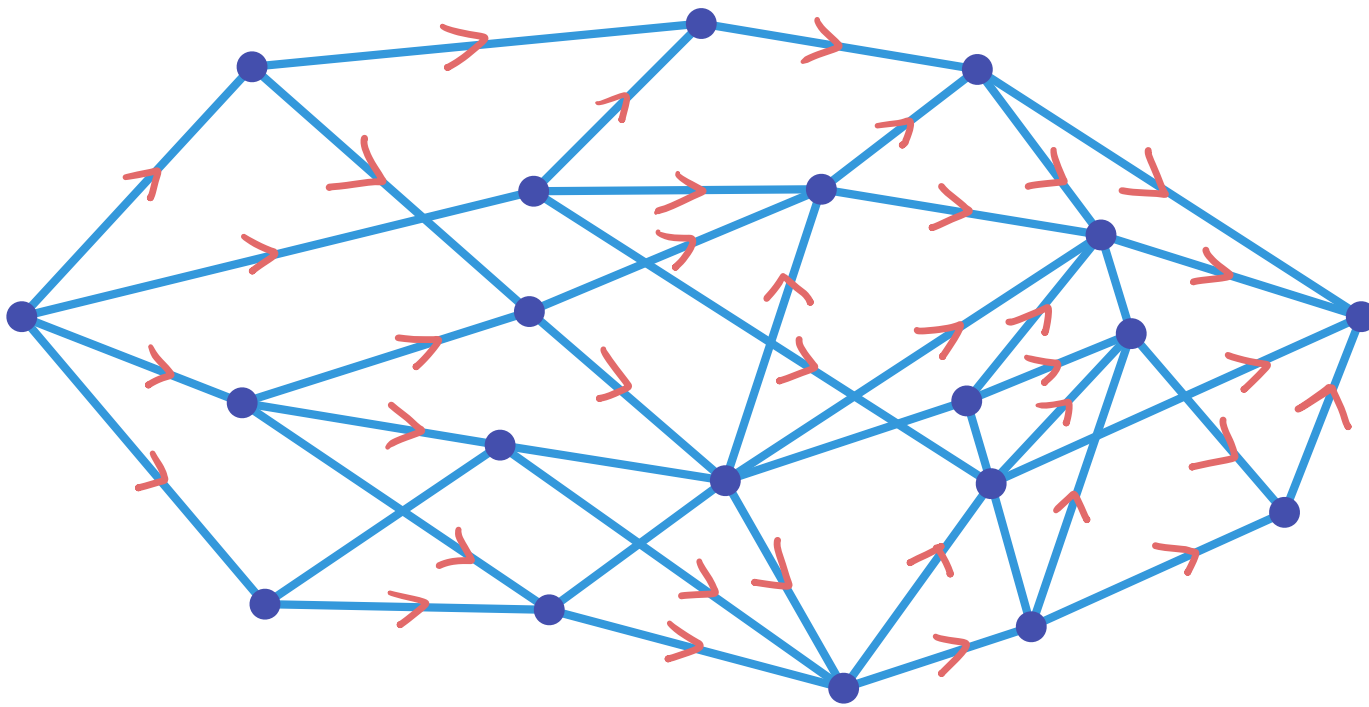
$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

Electrical flow = convex minimization

math setup

orient edges arbitrarily:



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization
math setup

Flow constraints = linear system

def $B: \mathbb{R}^E \xrightarrow{\text{(flows)}} \mathbb{R}^V$ by

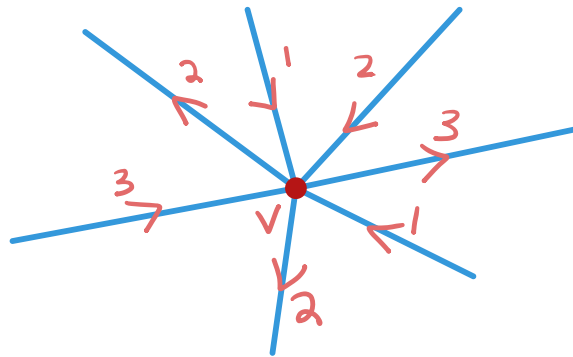
$$(Bf)_v = \text{net flow at } v$$

B is a linear function (check!)

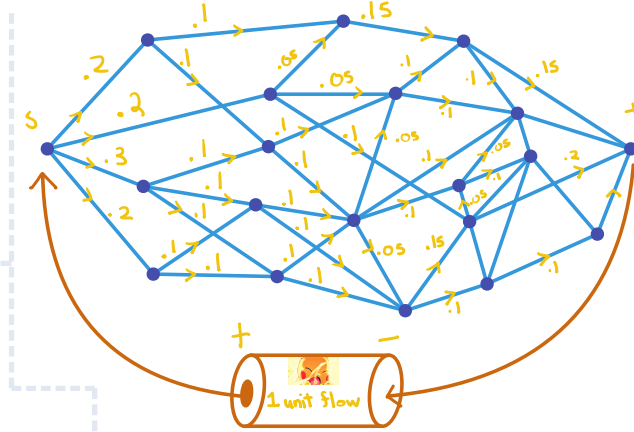
$\Rightarrow B$ is a matrix ($\mathbb{R}^{V \times E}$)

we want

$$Bf = d \quad \text{where} \quad d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$



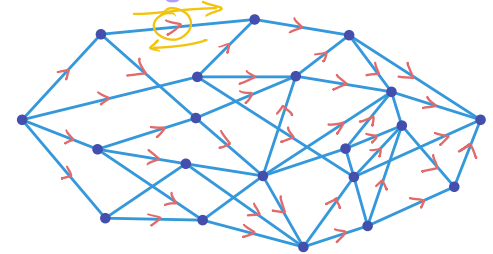
$$(Bf)_v = 0$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization

$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

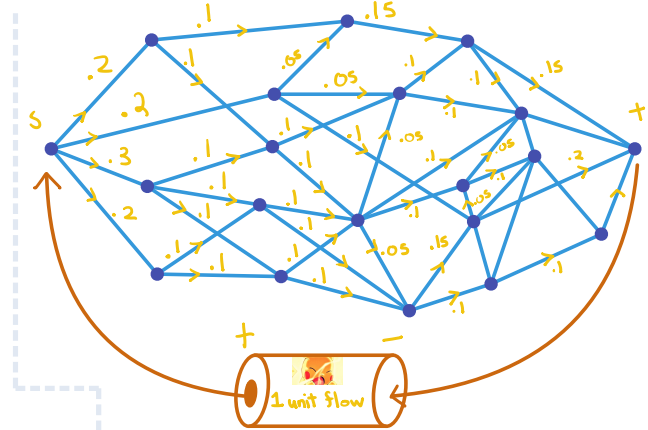
(electrical energy)

s.t. $Bf = d$

$(f \text{ is } (s,t)\text{-flow})$

electrical flow = optimum solution

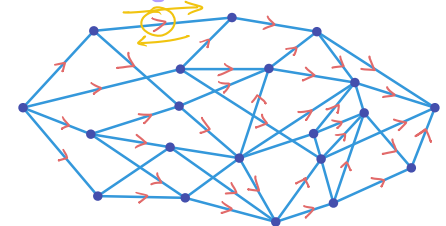
$\Rightarrow$ we should study (general)
optimality conditions



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def B: $\mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(BF)_v = \text{net flow at } v$$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

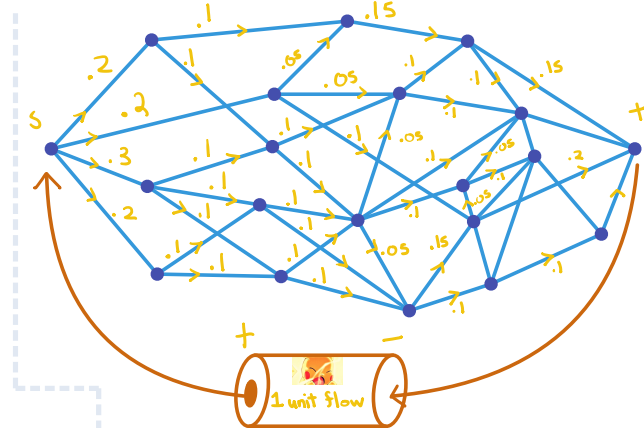
Lemma consider problem

$$\min f(x) \quad \text{s.t.} \quad Ax=b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(BF)_v = \text{net flow at } v$$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

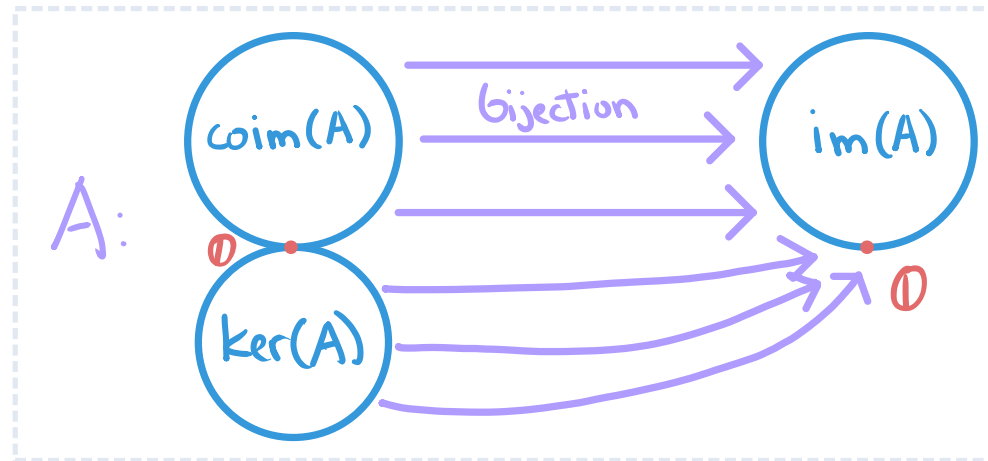
$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (f is (s)tr-Flow)

electrical flow = optimum solution

let $A: X \rightarrow Y$ linear map.

- $\text{kernel}(A) = \{x: Ax=0\}$
- $\text{image}(A) = \{Ax: x \in X\}$
- $\text{co-image}(A) =$ subspace orthogonal to $\text{ker}(A)$
 $= \{x: \langle x, z \rangle = 0 \ \forall z \in \text{ker}(A)\}$



Fact: A restricts to isomorphism between $\text{co-image}(A)$ and $\text{image}(A)$

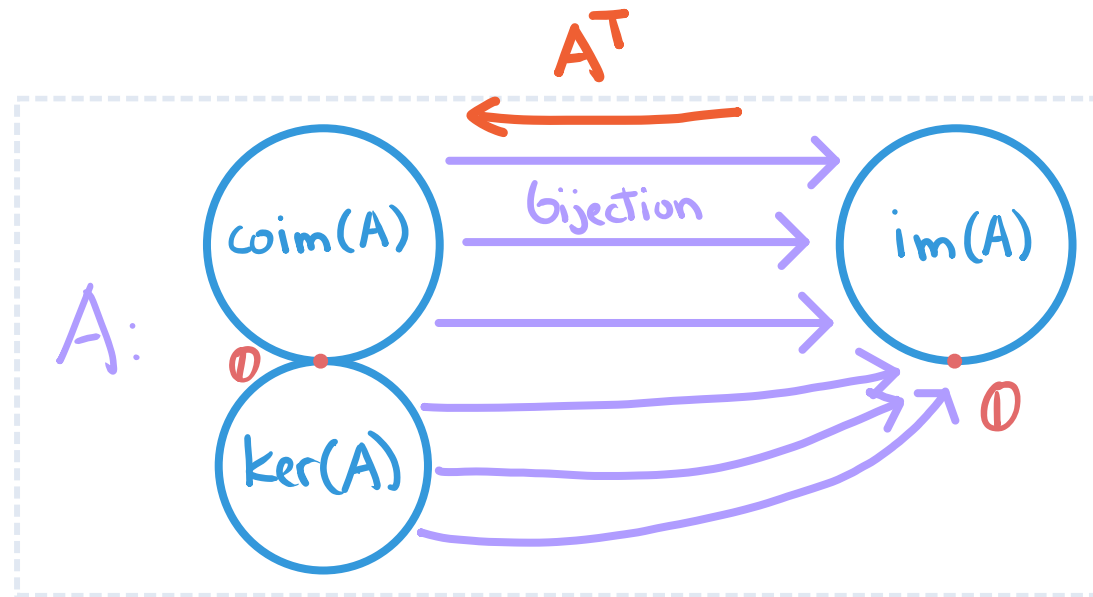
let $A: X \rightarrow Y$ linear map.

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- $\text{co-image}(A) = \text{subspace orthogonal to } \text{ker}(A)$
 $= \{x: \langle x, z \rangle = 0 \ \forall z \in \text{ker}(A)\}$

Fact:

$$\text{image}(A^T) = \text{co-image}(A)$$

$$\text{co-image}(A^T) = \text{image}(A)$$



Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

proof:

$$f'(x^*) \in \text{co-image}(A)$$

$$\text{i.e., } \langle f'(x^*), z \rangle = 0 \quad \forall \quad z: Az=0$$
$$< 0$$

$$f(x^* + \delta z) \approx f(x^*) + \delta \langle f'(x^*), z \rangle < f(x^*)$$

$$A(x^* + z) = Ax^* = b$$

let $A: X \rightarrow Y$ linear map.

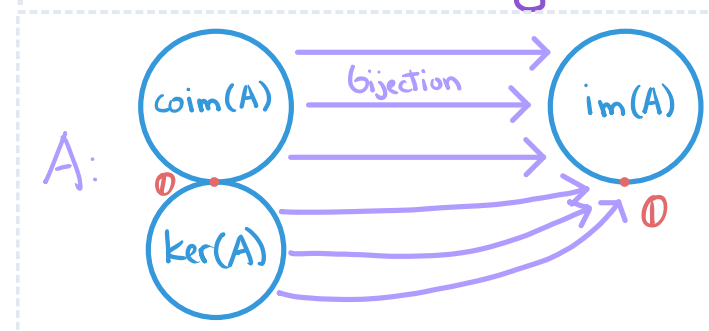
$$\bullet \text{ kernel}(A) = \{x: Ax=0\}$$

$$\bullet \text{ image}(A) = \{Ax: x \in X\}$$

$$\bullet \text{ co-image}(A) = \text{subspace orthogonal to } \text{ker}(A) \\ = \{x: \langle x, z \rangle = 0 \quad \forall z \in \text{ker}(A)\}$$

Fact: A restricts to isomorphism between $\text{co-image}(A)$ and $\text{image}(A)$

$$\text{Fact: } \text{image}(A^T) = \text{co-image}(A) \\ \text{co-image}(A^T) = \text{image}(A)$$



let $A: X \rightarrow Y$ linear map.

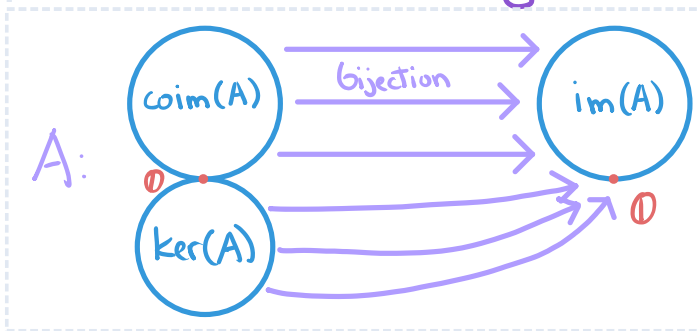
- $\text{kernel}(A) = \{x: Ax=0\}$

- $\text{image}(A) = \{Ax: x \in X\}$

- $\text{co-image}(A) = \text{subspace orthogonal to } \text{ker}(A)$
 $= \{x: \langle x, z \rangle = 0 \ \forall z \in \text{ker}(A)\}$

Fact: A restricts to isomorphism between
 $\text{co-image}(A)$ and $\text{image}(A)$

Fact: $\text{image}(A^T) = \text{co-image}(A)$
 $\text{co-image}(A^T) = \text{image}(A)$



Electrical flow = convex minimization

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2$$

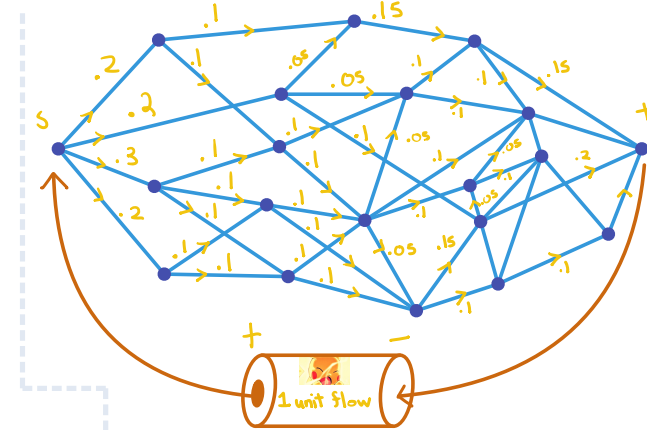
(electrical energy)

$$\text{s.t. } Bf = d$$

(f is (s,t) -flow)

electrical flow = optimum solution

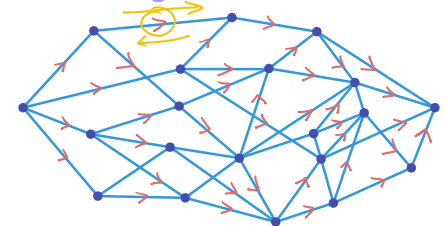
"effective" resistance of d = optimum value/energy



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

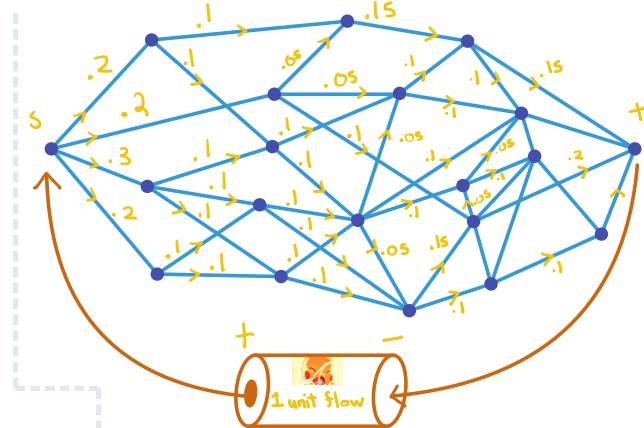
we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$
(vertex potentials)

$$R = \text{diag}(r(e_1), r(e_2), \dots, r(e_m))$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(Bf)_v = net flow at v

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

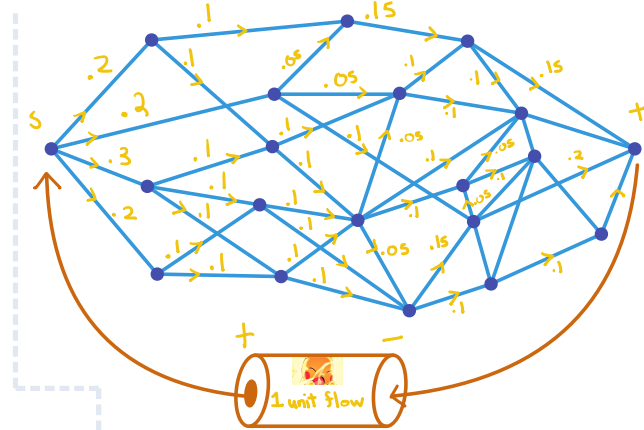
w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1} B^T p$
for some $p \in \mathbb{R}^V$
(vertex potentials)


$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (\$f\$ is (\$\pm\$)flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \quad \text{s.t.} \quad Ax=b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

Suppose f is electrical flow

$$\varphi(f) = \frac{1}{2} \sum_e r_e f_e^2 \quad \Big| \quad \begin{matrix} \text{"A"} \\ Bf = d \end{matrix}$$

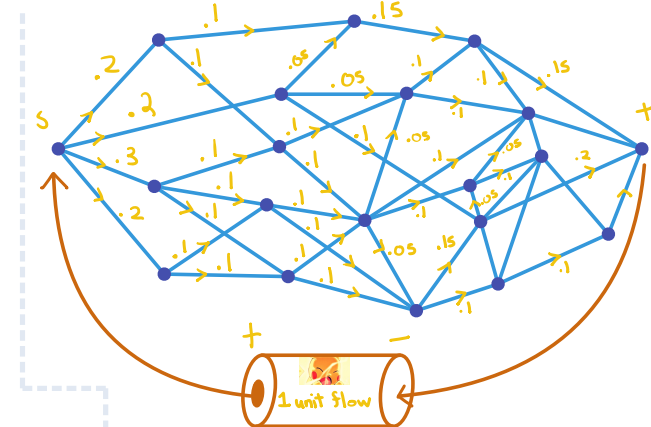
$$\varphi'_e(s) = r_e s_e$$

$$\varphi'(S) = RS$$

$$R\mathbf{s} = \mathbf{y}'(\mathbf{s}) = \mathbf{B}^T \mathbf{p} \Rightarrow \mathbf{s} = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{p}$$

Lemma consider problem

$$\frac{1}{2} \sum_e r_e f_e^2 = \frac{1}{2} \langle f, Rf \rangle$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax = b$$

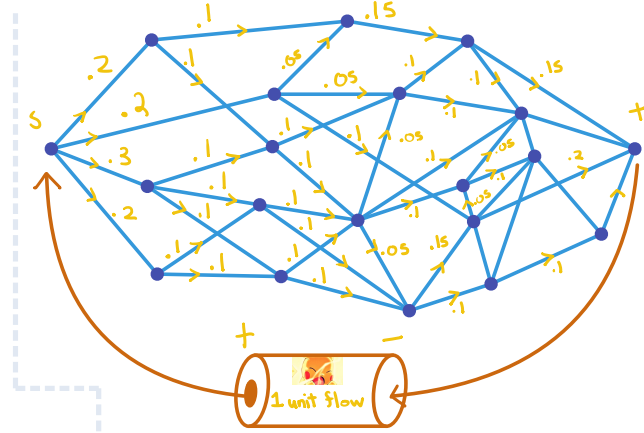
w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$
(vertex potentials)



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

Suppose $f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$

let $g: Bg=d$

$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y-x \rangle$ for convex φ

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(Bf)_v = net flow at v

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

$$\frac{1}{2} \langle g, Rg \rangle - \frac{1}{2} \langle f, Rf \rangle$$

$$\geq \langle Rf, g-f \rangle$$

$$= \langle R^{-1}B^T p, g-f \rangle$$

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

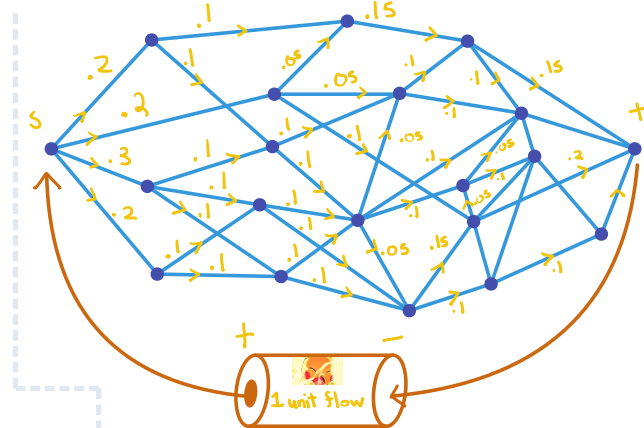
w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T \gamma \text{ for some } \gamma$$

$$= \langle p, B(y - f) \rangle$$

$$= \langle p, d - d \rangle = 0$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax = b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T \gamma \text{ for some } \gamma$$

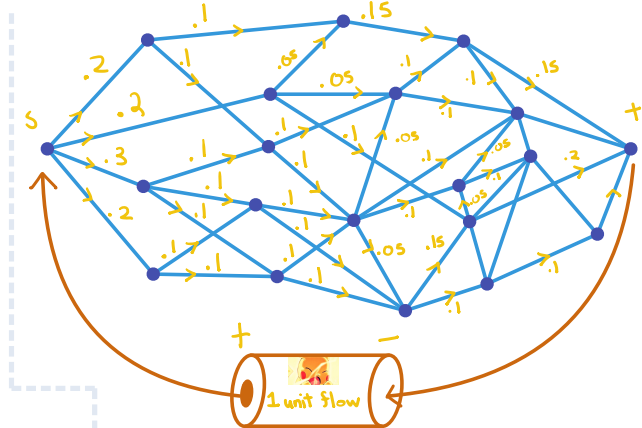
Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$ (vertex potentials)

$p \in \mathbb{R}^V$ called "electric potentials"

Ohm's law: $f = R^{-1}B^T p$

for electric flow f , electric potentials p



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t) -flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

Laplacian

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

"Laplacian" of graph w/ edge weights $w(e) = \frac{1}{r(e)}$

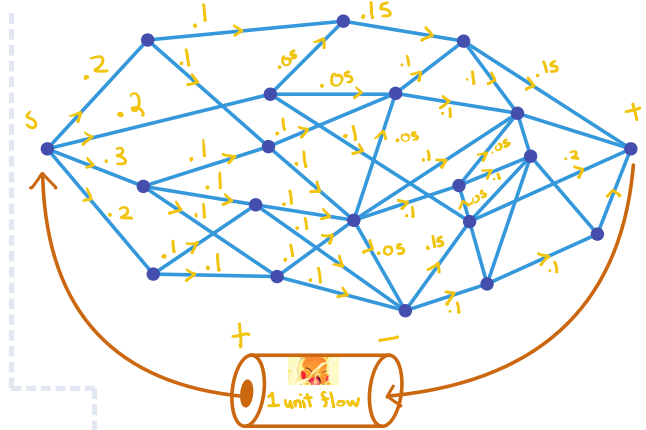
e.g.

xERV

$$\langle x, Lx \rangle = \langle B^T x, R^{-1} B^T x \rangle$$

$$= \sum_{e \in E} \frac{1}{r_e} (B^T x)_e^2$$

$$= \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$



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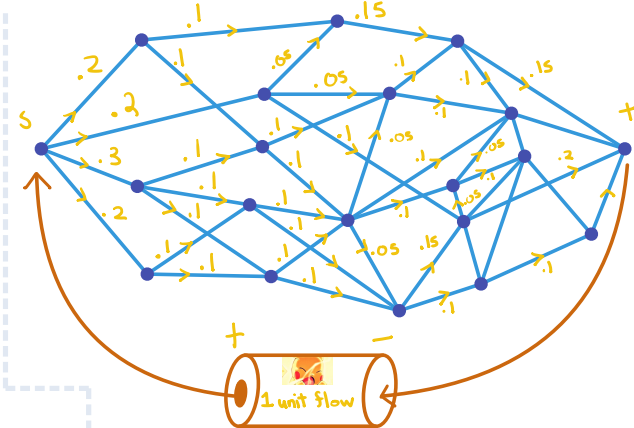
(electrical energy) (f is (s)t-slow)

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Ohm's law: $\xi = R^{-1} B^T p$
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"Laplacian" of graph w/ edge weights $w(e) = \frac{1}{r(e)}$

$$\checkmark \langle x, Lx \rangle = \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$

for $S \subseteq V$, let $x = \frac{1}{2}(\mathbb{1}_S - \mathbb{1}_{\bar{S}})$

$$\langle x, Lx \rangle = \sum_{\substack{e \in \partial(S) \\ e = \{u, v\}}} \frac{1}{r_e} \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right)^2$$

$$= \sum_{e \in \partial(S)} \frac{1}{r_e} = w(\partial(S))$$

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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Lemma

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

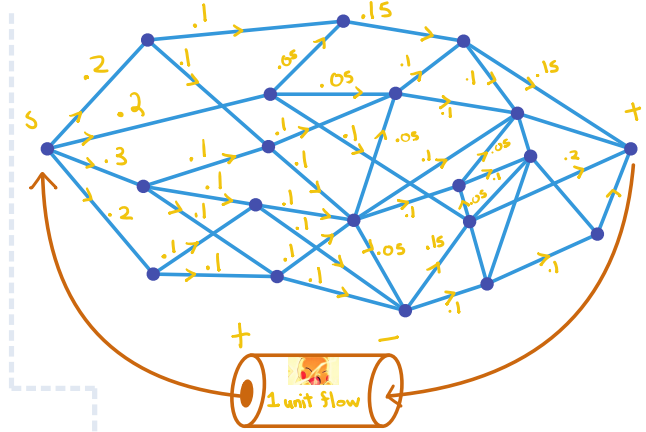
11

$$\min_{\mathbf{f}} \frac{1}{2} \langle \mathbf{f}, \mathbf{R} \mathbf{f} \rangle \quad \text{s.t.} \quad \mathbf{B} \mathbf{f} = \mathbf{d}$$

$$L = B R^T B^T$$

{ el. flow
p el. potentials } sorted

$$\begin{aligned}\langle \xi, R\xi \rangle &= \langle R^T B^T p, R R^{-1} B^T p \rangle \\ &= \langle p, B R^T B^T p \rangle \\ &= \langle p, Lp \rangle\end{aligned}$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

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(electrical energy) (S is (s)ti-Slow)

effective resistance of d = optimum value/energy

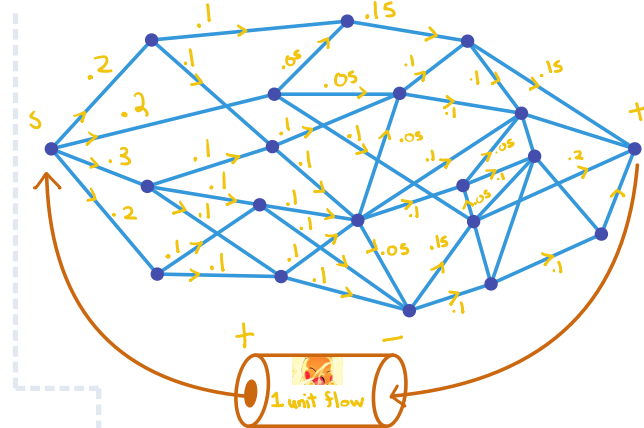
Ohm's law: $\xi = R^{-1} B^T p$
 for electric flow ξ , electric potentials p

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

$$= \langle L_P, L^{-1} L_P \rangle$$

$$= \langle d, L^{-1}d \rangle$$

$$L_P = B R^{-1} B^T = B S = d$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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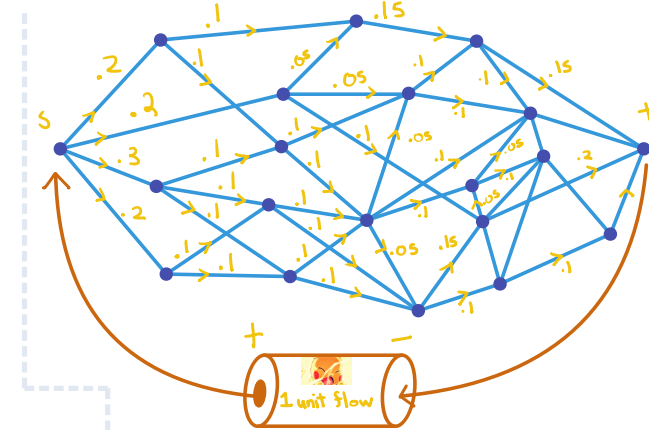
$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad Bf = d$$

(electrical energy) (f is (s)ti-(s)low)

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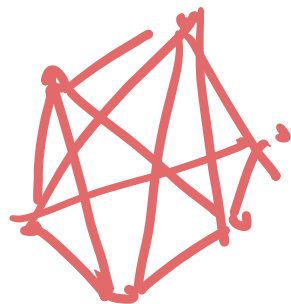
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(effective resist. of d) = $\langle d, L^{-1} d \rangle$

random walk terminology:

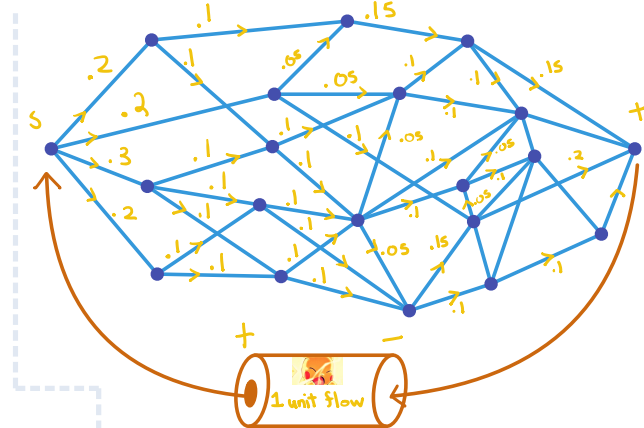
Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



$$K_n = O(n \log n)$$

"coupon collector"



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
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$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

$$H(t, t) = 0$$

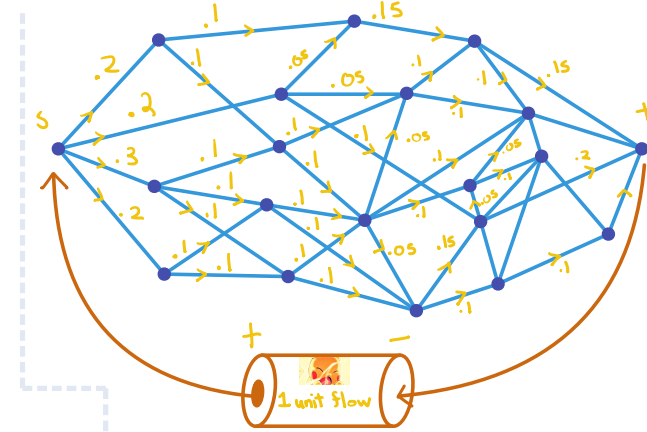
$$H(u, t) = 1 + \frac{1}{\deg(u)} \sum_{x \in N(u)} H(x, t)$$

$$p(u) = 1 + \frac{1}{\deg(u)} \sum_{v \in N(u)} p(v)$$

$$\deg(u)p(u) = \deg(u) + \sum_{v \sim u} p(v)$$

$$\sum_{v \in N(u)} p(u) - p(v) = \deg(u)$$

$$\sum_{v \in N(u)} f(v, u) = \deg(u)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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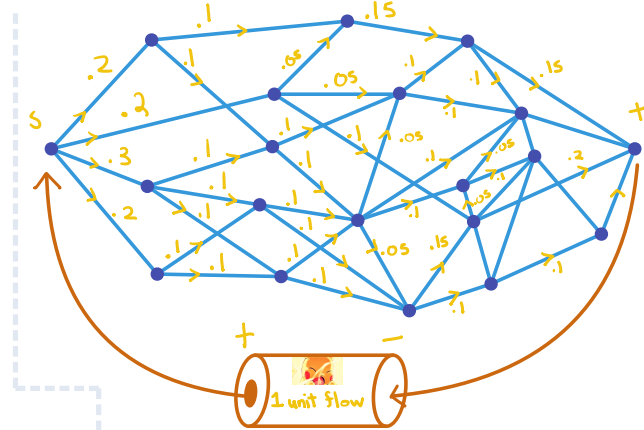
$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

net flow into u

$\Rightarrow$ net flow into $+$

$$= \sum_{u \neq t} \deg(u) = 2m - \deg(t)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

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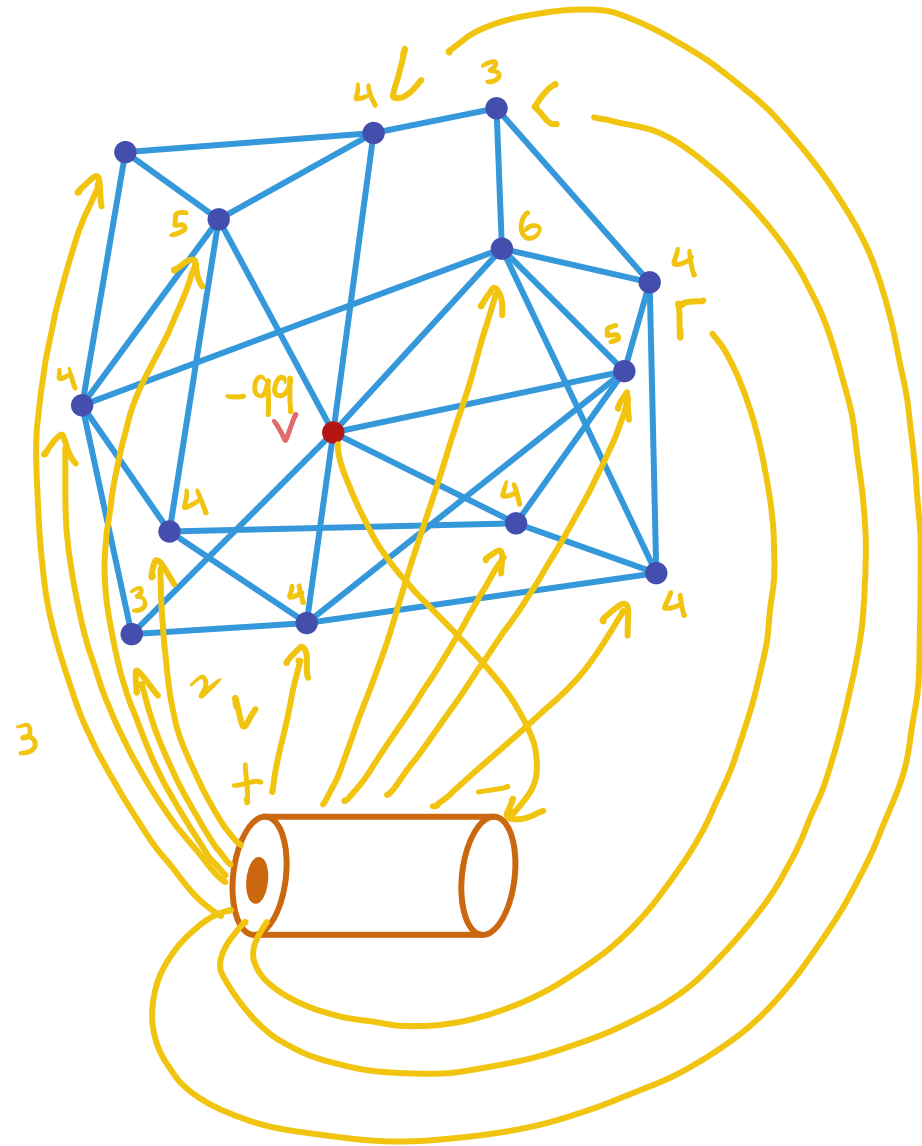
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$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

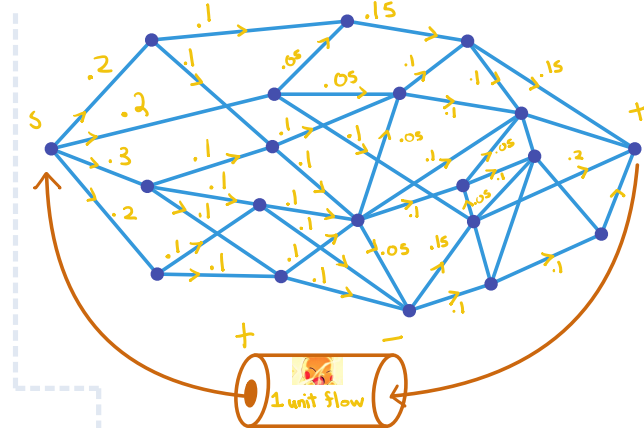
$\Rightarrow p$ encodes flow routing

$$d_v = \begin{cases} \deg(u) & v \neq t \\ -(2m - \deg(t)) & v = t \end{cases}$$



Lemma

$$H(u,v) + H(v,u) = 2m \left(\begin{array}{c} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

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def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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Lemma

$$H(u,v) + H(v,u) = 2m \left(\begin{array}{l} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$

proof: let $p = H(\cdot, v)$

$$q = H(\cdot, u)$$

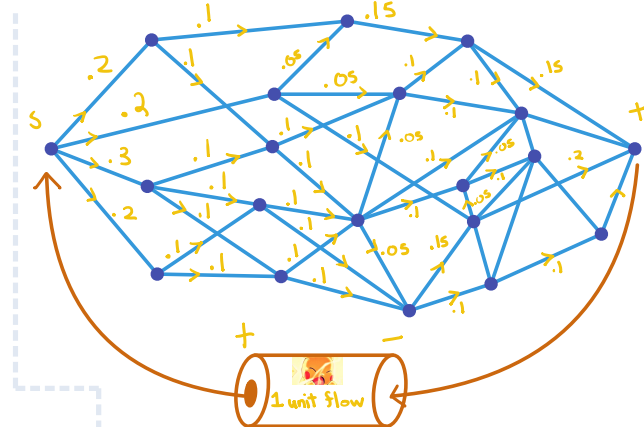
$$r = p - q$$

$$H(u,v) + H(v,u) = p(u) + q(v)$$

$$= (p(u) - p(v)) + (q(v) - q(u))$$

$$= r(u) - r(v)$$

$$d_x = \begin{cases} 1 & x = u \\ -1 & x = v \\ 0 & \text{otherwise} \end{cases} \quad \leftarrow$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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(electrical energy) (f is s(t)-flow)

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Ohm's law: $f = R^{-1} B^T p$
for electric flow f , electric potentials p

$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

$$r_n - r_v = \langle d, r \rangle$$

$$r = p - q$$

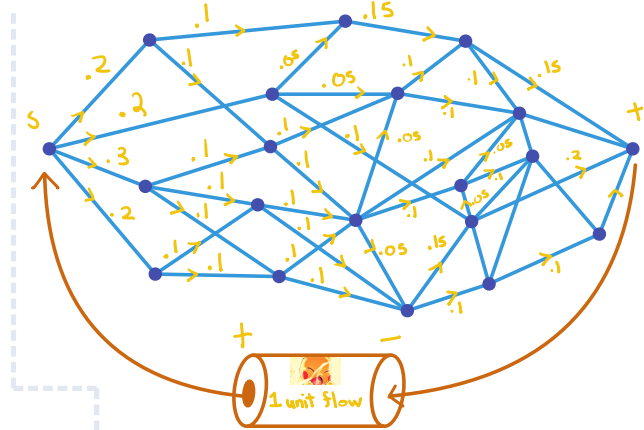
$$B^T r = B^T p - B^T a$$

Slows

routes 2m units slow
from u to v

$$BB^T r = 2 \bmod$$

$$B^T B \frac{r}{2m} = d$$



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$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1}d \rangle$$

$\frac{1}{2m} r = \text{electr. pot. for } d$

$$H(u, v) + H(v, u) = p_u - q_v$$

$$= \langle d, r \rangle$$

$$= \frac{1}{2m} \langle B B^T r, r \rangle$$

$$= \frac{1}{2m} \langle B^T r, B^T r \rangle$$

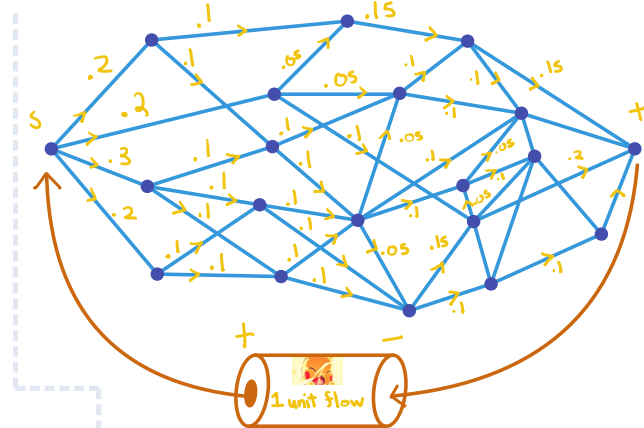
$$= 2m \left\langle \frac{B^T r}{2m}, \frac{B^T r}{2m} \right\rangle = 2m (\text{elect. energy})$$

$1 \text{ } u \rightarrow v$

random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

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$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

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$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

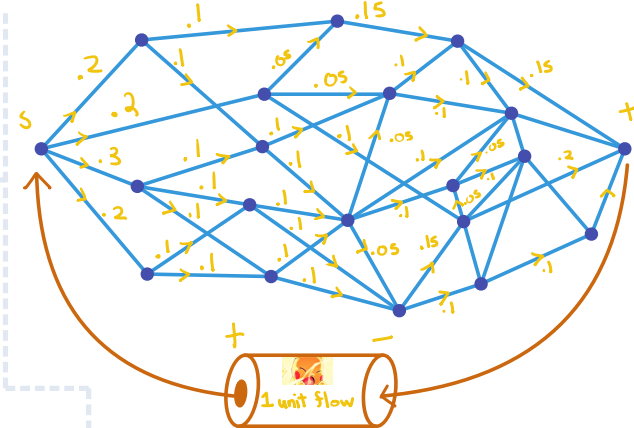
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$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

Lemma $H(u,v) + H(v,u) = 2m$ (energy of unit (u,v) -electric flow)

Theorem $C(s) \leq 2mn$



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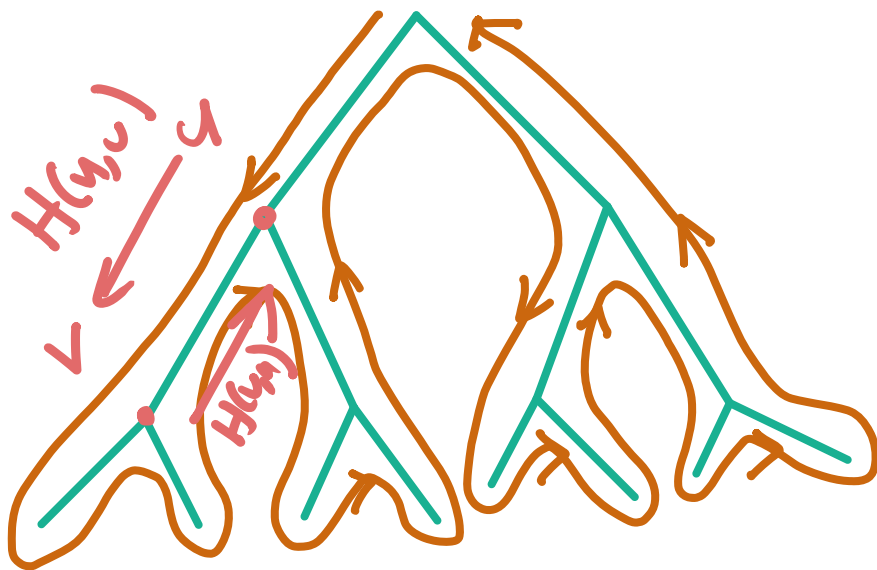
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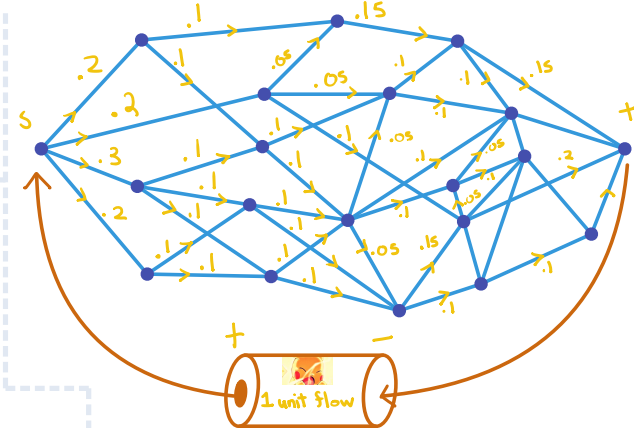
Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\text{total time} = \sum_{\{u,v\} \in T} H(u,v) + H(v,u)$$

$$= 2m \sum_{\{u,v\} \in T} (\text{energy of unit } (u,v)\text{-electric flow})$$



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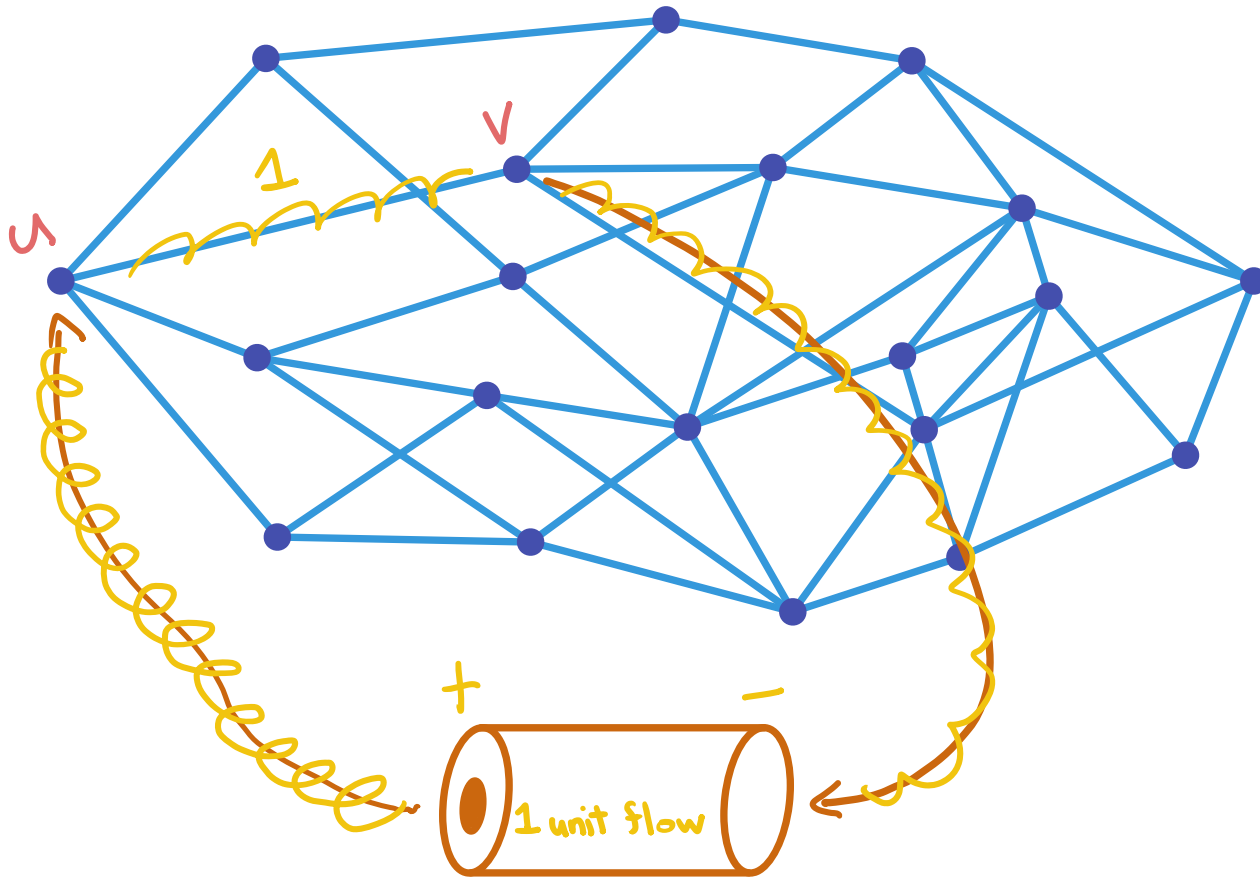
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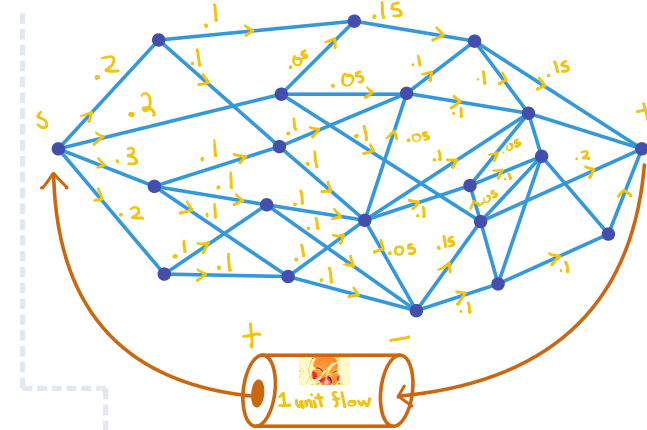
$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

$$\text{total time} = 2m \sum_{\{u,v\} \in E} (\text{energy of unit } (u,v)\text{-electric flow})$$



$$(\text{energy of unit } (u,v)\text{-electric flow}) \leq$$

1



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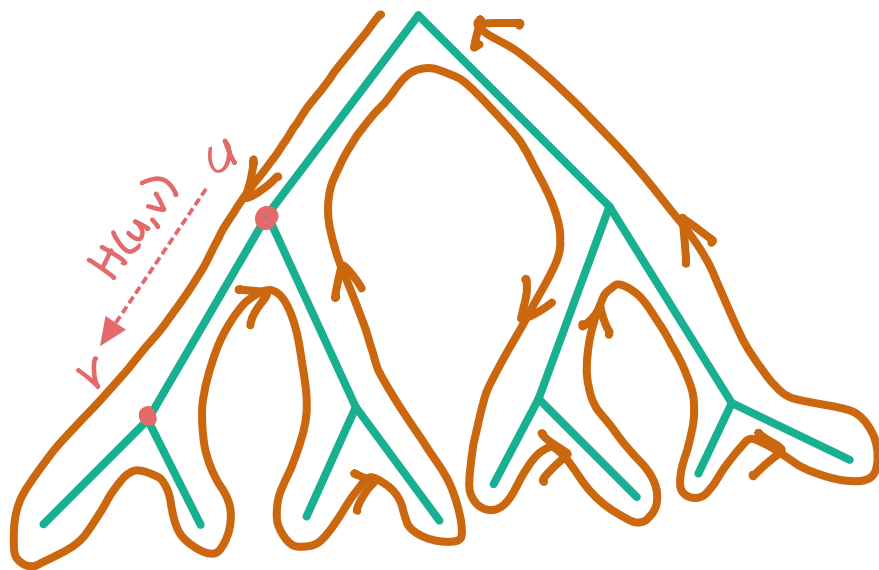
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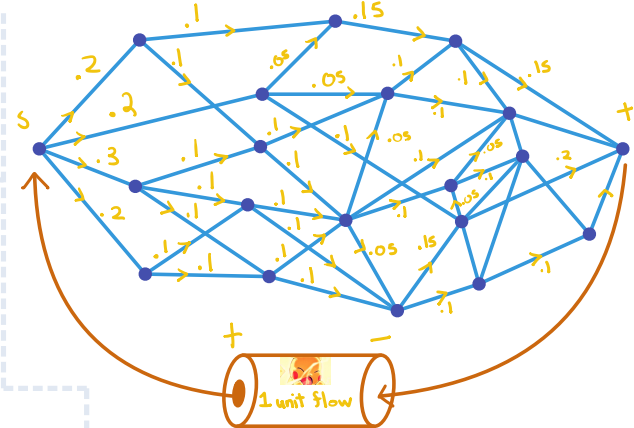
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proof Six traversal of spanning tree



$$\text{total time} = \sum_{\{u,v\} \in T} H(u,v) + H(v,u)$$

$$= 2m \sum_{\{u,v\} \in T} (\text{energy of unit } (u,v)\text{-electric flow}) \leq 2m(n-1)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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Ohm's law: $f = R^{-1} B^T p$
 for electric flow f, electric potentials p

$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

(effective resist. of d) = $\langle d, L^{-1} d \rangle$